Vector Analysis
Spring 2014

Exercise 7

Recital on Mon 5.5.

1. a) Show that the volume of an object is given by the formula

$$V = \frac{1}{3} \oiint_S \vec{r} \cdot \hat{N} dS,$$

where $\vec{r}$ is the position vector and S is the surface of the object.

b) Using the law of Gauss that

$$\iiint_D \nabla \phi dV = \oiint_S \phi \hat{N} dS.$$

Hint: Apply the Gauss law to the vector $\vec{F} = \phi \vec{c}$, where $\vec{c}$ is an arbitrary constant vector.

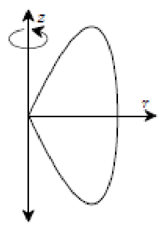
2. Evaluate the surface integral $\oiint_D (2xy\hat{i} + 3y\hat{j} + 2z\hat{k}) \cdot d\vec{S}$, where D is the region bounded by the coordinate planes $x = 0$, $y = 0$ ja $z = 0$ and the plane $x + y + z = 1$.

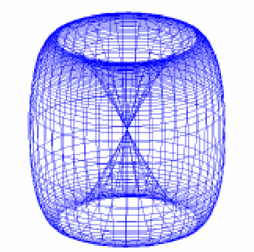
(Answer: 11/12.)

3. Consider an object obtained when the curve

$$r = \sqrt{x^2 + y^2} = \cos u,$$

$$z = \sin 2u, \qquad -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}.$$





rotates around the *z*-axis.

Calculate the volume of the object,

$$V = \iiint_{\text{kappale}} dV$$

using Gauss' law.

Hints: Notice that the simplest choice for the vector field is $\vec{F} = z\hat{k}$. To calculate the surface integral, give a parameter representation of the surface in

terms of u and the rotation angle θ $(0 \le \theta \le 2\pi)$. Present the surface unit dS in terms of $dud\theta$.
(Answer: $\pi^2/2$.)

4. Consider the unit sphere $x^2 + y^2 + z^2 \le 1$. Use Gauss' law to evaluate

$$I = \iiint_D z^2 dV.$$

Hint: Again it would be wise to choose the simplest possible vector field.
(Answer: $4\pi/15$.)

5. Consider the vector field

$$\vec{F} = (z-2y)\hat{i} + (3x-4y)\hat{j} + (z+3y)\hat{k}.$$

Using Stokes' theorem to evaluate

$$\oint_C \vec{F} \cdot d\vec{r},$$

where the curve C is

a) a unit sphere on the plane z = 2,

b) the boundary of the triangle, whose vertices are at (1,0,0), (0,1,0) and (0,0,1).

(Answers 5π or -5π ja $9/2$ or -9/2.)

6. Calculate the curve integral of

$$\vec{F} = (e^x - y^3)\hat{i} + (e^y + x^3)\hat{j} + e^z\hat{k}$$

along the curve

$$\vec{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + \sin 2t\,\hat{k} \qquad (0 \le t \le 2\pi).$$

Hint: Stokes. Notice that the curve is on the plane $z = 2xy$, according to a well known trigonometric rule.
(Answer: $3\pi/2$ or -3π/2.)