Vektorianalyysi
k. 2014

Harjoitus 4

Käsitellään ke 2.4.

1. Olkoon pinta *S* kolmio, jonka kärjet ovat (*x*,*y*)-tason pisteissä (0,0), (1,0) ja (1,1).

    a. Laske kolmion pinta-ala integroimalla.
    b. Laske

$$\iint_S dS\ \frac{\sin x}{x}.$$

2. Tarkastellaan integraalia

$$\iint_A dA\ y^3\,\frac{e^{x^2}}{x} = \int_{y=0}^{2}\int_{x=y^2}^{4} dxdy\ y^3\,\frac{e^{x^2}}{x}.$$

    a. Piirrä pinta *A*.
    b. Suorita integrointi.

3. Olkoon integroimisalueena *S* tasopinta $S=\{(x,y): x+y\geq 0, y\leq 0, x\leq 1\}$.

    a. Tee muuttujanvaihto $u=x+y,\ v=x$ ja laske vastaava Jacobin determinantti.
    b. Mikä on *S*:ää vastaava integroimisalue (*u*,*v*)-tasossa?
    c. Laske pintaintegraali

$$\iint_S dS\ x^3\sqrt{x+y}$$

    em. muuttujanvaihtoa käyttäen.

4. Laske integraali

$$\iint_S dS\ (x+2y+3z).$$

Tässä *S* on se tason $2x-y+z=3$ osa, joka on *xy*-tasossa olevan, *x*- ja *y*-akselin ja suoran $y=1-2x$ rajaaman kolmion yläpuolella.

5. Laske pintaintegraali

$$\iint_S dS\ (x^2 + y^2 + 3z^2),$$

jossa S on se osa paraboloidin $z = x^2 + y^2$ pintaa, jossa $x^2 + y^2 \le 9$.

6. Laske pintaintegraali

$$\iint_S dxdy\ e^{-x^2-y^2}, \quad a^2 \le x^2 + y^2 \le b^2$$

sopivalla (ja ilmeisellä) muuttujanvaihdolla. Piirrä alue S.

Päättele saamasi tuloksen perusteella integraalin $\int_0^\infty dx\ \mathrm{e}^{-x^2}$ arvo.

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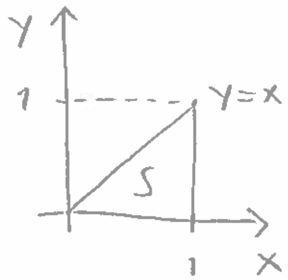
$\boxed{1.}$



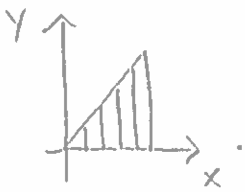
Käytän jatkossa vain yhtä integraalimerkkiä

a.) Lasketaan ensin kolmion pinta-ala.

$$S = \{0 \le x \le 1,\ 0 \le y \le x\}$$

$$\int_S dA = \int_0^1 dx \int_0^x dy = \int_0^1 dx\, x = \Big/_0^1 \tfrac{1}{2}x^2 = \tfrac{1}{2}$$

Tämä vastasi "siivutusta" x-suunnassa



Tämä voidaan laskea myös integroimalla x-suunta ensin.

$$\int_S dA = \int_0^1 dy \int_y^1 dx = \int_0^1 dy\,(1-y) = \Big/_0^1 y - \tfrac{1}{2}y^2 = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

b.) $$\int_S dA \frac{\sin x}{x} = \int_0^1 dx \int_0^x dy \frac{\sin x}{x} = \int_0^1 dx \frac{\sin x}{x} \cdot x = \int_0^1 dx \sin x$$

$$= -\Big/_0^1 \cos x = -\cos 1 + 1$$

<u>Huom.</u> integrointi ei onnistu toisessa järjestyksessä

$$\int_0^1 dy \int_y^1 dx \frac{\sin x}{x}.$$

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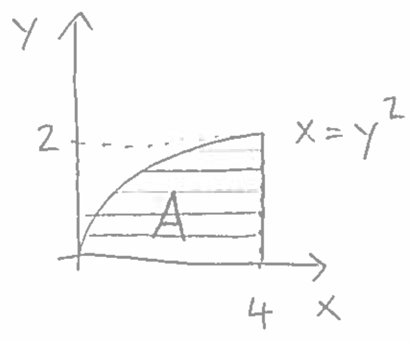
$\boxed{2.}$

$$\int_A dA\, y^3 \frac{e^{x^2}}{x} = \int_0^2 dy \int_{y^2}^4 dx\, y^3 \frac{e^{x^2}}{x}$$

a.)



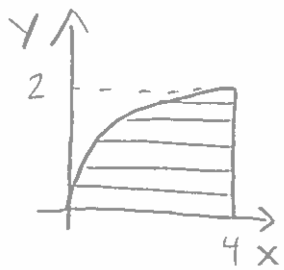
b.) Integrointi ei onnistu annetussa järjestyksessä, joten vaihdetaan rajat.

$$x = y^2 \rightarrow y = \sqrt{x}$$

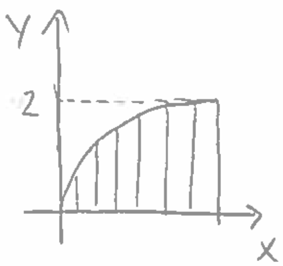
$$\int_A dA\, y^3 \frac{e^{x^2}}{x} = \int_0^4 dx \int_0^{\sqrt{x}} dy\, y^3 \frac{e^{x^2}}{x} = \int_0^4 dx \Big/_0^{\sqrt{x}} \frac{1}{4} y^4 \frac{e^{x^2}}{x}$$

$$= \frac{1}{4}\int_0^4 dx\, x^2 \frac{e^{x^2}}{x} = \frac{1}{4}\int_0^4 dx\, x e^{x^2} = \frac{1}{4} \Big/_0^4 \frac{1}{2} e^{x^2} = \frac{1}{8}\left(e^{16} - 1\right)$$

Huom. Ennen integrointirajojen vaihtoa "siivutettiin" y-suunnassa ja vaihdon jälkeen x-suunnassa.



Ennen



Jälkeen

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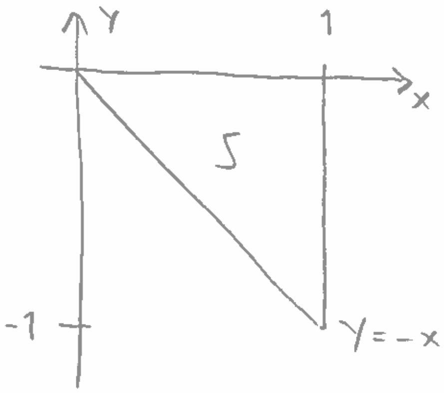
$\boxed{3.}$ $S = \{(x,y) \in \mathbb{R}^2,\ x+y \geq 0,\ y \leq 0,\ x \leq 1\}$



a.) Tehdään muuttujanvaihto

$$\begin{cases} u = x+y \\ v = x \end{cases} \Rightarrow \begin{cases} x = v \\ y = u - x = u - v \end{cases}$$

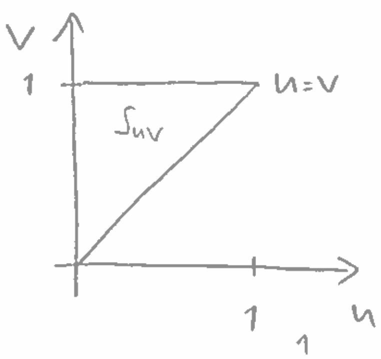
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = \underline{\underline{-1}}$$

b.) Tutkitaan miten pinnan S määrittelevät ehdot muuntuvat koordinaattimuunnoksessa.

$x + y \geq 0 \Rightarrow u \geq 0$

$y \leq 0 \Rightarrow u - v \leq 0 \Rightarrow u \leq v$

$x \leq 1 \Rightarrow v \leq 1$



$\Rightarrow S_{uv} = \{(u,v) \in \mathbb{R}^2,\ u \geq 0,\ u \leq v,\ v \leq 1\}$

c.) $$\int_S ds\, x^3\sqrt{x+y} = \int_0^1 dv \int_0^v du \overbrace{1}^{|J|} \cdot v^3\sqrt{u} = \int_0^1 dv \Big/_0^v v^3 \cdot \frac{2}{3} u^{3/2}$$

$$= \frac{2}{3}\int_0^1 dv\, v^{9/2} = \frac{2}{3} \Big/_0^1 \frac{2}{11} v^{11/2} = \underline{\underline{\frac{4}{33}}}$$

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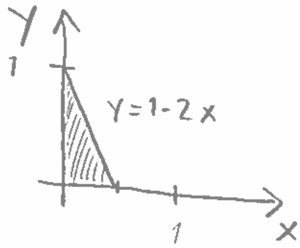
4.

$$\int_S ds\,(x+2y+3z)$$



$$S=\left\{(x,y,z)\in\mathbb{R}^3,\ 2x-y+z=3,\ 0\le y\le 1-2x\ ,\ 0\le x\le \tfrac{1}{2}\right\}$$

Parametrisoidaan pinta $x=u,\ y=v,\ z=3-2u+v$

$$\Rightarrow \vec{r}(u,v)=u\hat{\imath}+v\hat{\jmath}+(3-2u+v)\hat{k}\ ,\quad 0\le v\le 1-2u\ ,\ 0\le u\le \tfrac{1}{2}$$

$$ds=\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|du\,dv$$

$$\frac{\partial\vec{r}}{\partial u}=\hat{\imath}-2\hat{k}\ ,\quad \frac{\partial\vec{r}}{\partial v}=\hat{\jmath}+\hat{k}\ ,\quad \frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}=\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\1&0&-2\\0&1&1\end{vmatrix}=2\hat{\imath}-\hat{\jmath}+\hat{k}$$

$$\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|=\sqrt{2^2+1+1}=\sqrt{6}$$

$$\Rightarrow \int_S ds\,(x+2y+3z)=\int_0^{1/2}du\int_0^{1-2u}dv\,\sqrt{6}\,\big(\underbrace{u+2v+3(3-2u+v)}_{u+2v+9-6u+3v\,=\,9-5u+5v}\big)$$

$$=\sqrt{6}\int_0^{1/2}du\,\Big/_0^{1-2u}\,9v-5uv+\tfrac{5}{2}v^2=\sqrt{6}\int_0^{1/2}du\left(9(1-2u)-5u(1-2u)+\tfrac{5}{2}(1-2u)^2\right)$$

$$=\sqrt{6}\int_0^{1/2}du\left(9-18u-5u+10u^2+\tfrac{5}{2}-10u-10u^2\right)$$

$$=\sqrt{6}\int_0^{1/2}du\left(\frac{23}{2}-33u+20u^2\right)=\sqrt{6}\,\Big/_0^{1/2}\,\frac{23}{2}u-\frac{33}{2}u^2+\frac{20}{3}u^3$$

$$=\sqrt{6}\left(\frac{23}{4}-\frac{33}{8}+\frac{20}{24}\right)=\sqrt{6}\left(\frac{138}{24}-\frac{99}{24}+\frac{20}{24}\right)=\underline{\underline{\frac{59}{24}\sqrt{6}}}$$

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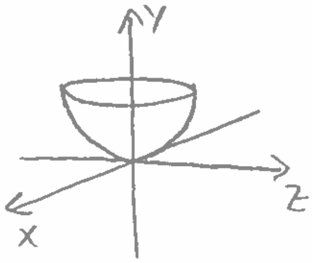
$\boxed{5.}$ Laske $\int_S ds\,(x^2+y^2+3z^2)$ kun

$S = \{(x,y,z) \in \mathbb{R}^3,\ z = x^2+y^2,\ \underbrace{x^2+y^2 \le 9}\}$

Tämä on ympyrä xy-tasossa



$\Rightarrow x = \rho\cos\phi,\quad y = \rho\sin\phi,\quad \rho \in [0,3]$
$\phi \in [0, 2\pi[$

$\Rightarrow z = \rho^2\cos^2\phi + \rho^2\sin^2\phi = \rho^2$

Pinnan parametriesitys on siis:

$\vec{r}(\rho,\phi) = \rho\cos\phi\,\hat{\imath} + \rho\sin\phi\,\hat{\jmath} + \rho^2\hat{k},\quad \rho \in [0,3],\ \phi \in [0,2\pi[$

$\frac{\partial \vec{r}}{\partial \rho} = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath} + 2\rho\hat{k},\quad \frac{\partial \vec{r}}{\partial \phi} = -\rho\sin\phi\,\hat{\imath} + \rho\cos\phi\,\hat{\jmath}$

$$\frac{\partial \vec{r}}{\partial \rho} \times \frac{\partial \vec{r}}{\partial \phi} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\phi & \sin\phi & 2\rho \\ -\rho\sin\phi & \rho\cos\phi & 0 \end{vmatrix} = \hat{\imath}(0 - 2\rho^2\cos\phi) - \hat{\jmath}(+2\rho^2\sin\phi) + \rho\hat{k}$$

Yleisessä parametriesityksessä on:

$$ds = \left|\frac{\partial \vec{r}}{\partial \rho} \times \frac{\partial \vec{r}}{\partial \phi}\right| d\rho\, d\phi = \sqrt{4\rho^4\cos^2\phi + 4\rho^4\sin^2\phi + \rho^2}\, d\rho\, d\phi$$
$$= \rho\sqrt{4\rho^2+1}\, d\rho\, d\phi$$

$$\Rightarrow \int_S ds\,(x^2+y^2+3z^2) = \int_0^3 d\rho \int_0^{2\pi} d\phi\, \rho\sqrt{4\rho^2+1}\,(\rho^2 + 3\rho^4)$$

$$= 2\pi \Big|_0^3 \frac{1}{420}(4\rho^2+1)^{3/2}(45\rho^4 + 12\rho^2 - 2)$$

↑ Wolfram α. Jos W ei ole käytössä, niin osittaisintegroi pari kertaa.

$$= \frac{\pi}{210}\left[37^{3/2}(3645 + 108 - 2) + 2\right] = \frac{\pi}{210}\left(3751 \cdot 37^{3/2} + 2\right)$$

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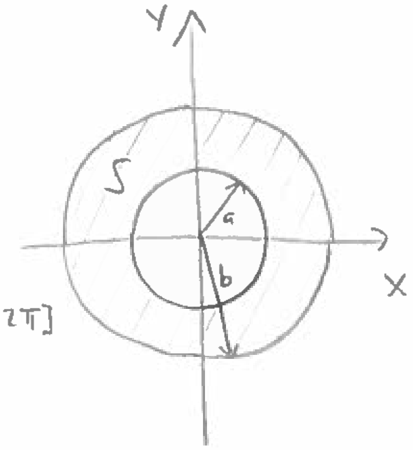
$\boxed{6.}$ $\int_S ds\, e^{-x^2-y^2}$, $a^2 \leq x^2+y^2 \leq b^2$



Parametrisoidaan pinta S

$\vec{r} = \rho\cos\phi\,\hat{\imath} + \rho\sin\phi\,\hat{\jmath}$, $\rho \in [a,b]$, $\phi \in [0,2\pi]$

$$\frac{\partial \vec{r}}{\partial \rho} = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}$$

$$\frac{\partial \vec{r}}{\partial \phi} = -\rho\sin\phi\,\hat{\imath} + \rho\cos\phi\,\hat{\jmath}$$

$$\frac{\partial \vec{r}}{\partial \rho} \times \frac{\partial \vec{r}}{\partial \phi} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \end{vmatrix} = \hat{k}(\rho\cos^2\phi + \rho\sin^2\phi) = \rho\hat{k}$$

$$\Rightarrow ds = \rho\, d\rho\, d\phi$$

$$\Rightarrow \int_S ds\, e^{-x^2-y^2} = \int_a^b d\rho \int_0^{2\pi} d\phi\, \rho\, e^{-\rho^2} = 2\pi \int_a^b d\rho\, \rho\, e^{-\rho^2}$$

$$= 2\pi \Big/_a^b -\tfrac{1}{2}e^{-\rho^2} = \underline{\pi\left(e^{-a^2} - e^{-b^2}\right)}$$

$$\int_0^\infty dx\, e^{-x^2} = \frac{1}{2}\int_{-\infty}^{\infty} dx\, e^{-x^2} = \frac{1}{2}\left(\left(\int_{-\infty}^{\infty} dx\, e^{-x^2}\right)\left(\int_{-\infty}^{\infty} dy\, e^{-y^2}\right)\right)^{1/2}$$

$$= \frac{1}{2}\Bigg(\underbrace{\int_{\mathbb{R}^2} dx\,dy\; e^{-x^2-y^2}}_{}\Bigg)^{1/2} = \frac{1}{2}(\pi)^{1/2} = \frac{\sqrt{\pi}}{2}$$

Tämä on aiemmin laskettu integraali kun $a=0$ ja $b \to \infty$