Vektorianalyysi
k. 2014

Harjoitus 5

Käsitellään ke 9.4.

1. Mitä geometrisia objekteja *xyz*-avaruudessa vastaavat ne pisteet, joiden pallokoordinaateille pätee
   a) $r=$ vakio, b) $\phi=$ vakio, c) $\theta=$ vakio, d) $r=$ vakio ja $\phi=$ vakio,
   e) $r=$ vakio ja $\theta=$ vakio, f) $\phi=$ vakio ja $\theta=$ vakio,
   g) $r=$ vakio, $\phi=$ vakio ja $\theta=$ vakio?

2. Tarkastellaan avaruuden pistettä *P*, jonka koordinaatit ovat $(x,y,z)=(1,1,1)$. Mitkä ovat tähän pisteeseen asetettujen paikallisten sylinteri- ja pallokoordinaatistojen kantavektorit $\hat{\rho},\hat{\phi},\hat{z}$ ja $\hat{r},\hat{\theta},\hat{\phi}$ *xyz*-koordinaatiston kantavektoreiden $\hat{i},\hat{j}$ ja $\hat{k}$ avulla lausuttuina?

3. Johda napakoordinaatistossa skalaarifunktion $u(r,\theta)$ gradientti ja vektorifunktion $\vec{v}=v_r(r,\theta)\hat{r}+v_\theta(r,\theta)\hat{\theta}$ divergenssi $\nabla\cdot\vec{v}$.

4. Osoita, että sylinterikoordinaatistossa

$$\nabla^2 u(r,\phi,z)=\frac{\partial^2 u}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2}+\frac{\partial^2 u}{\partial z^2}$$

   lähtemällä Laplacen operaattorin karteesisesta lausekkeesta $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

   ja soveltamalla derivoinnin ketjusääntöä.

5. Johda vektorikentän $\vec{F}=F_r\hat{r}+F_\theta\hat{\theta}+F_\phi\hat{\phi}$ divergenssi $\nabla\cdot\vec{F}$ pallokoordinaatistossa eli

$$\nabla\cdot\vec{F}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_\theta\sin\theta)+\frac{1}{\rho\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

   osittaisderivointia käyttäen lähtemällä muodosta (ks. luennot s. 66)

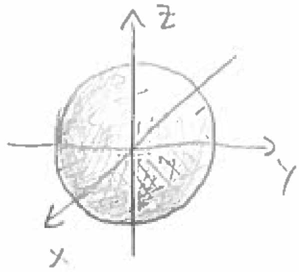
$$\nabla\cdot\vec{F}=(\hat{i}\partial_x+\hat{j}\partial_y+\hat{k}\partial_z)\cdot(F_r\hat{r}+F_\theta\hat{\theta}+F_\phi\hat{\phi}).$$

FYSA114 Demo 5

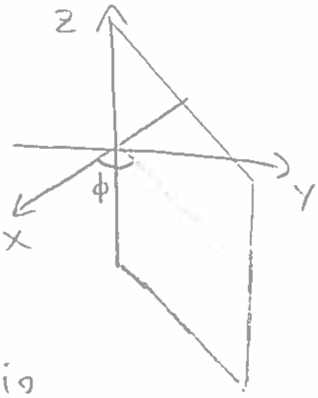
1.

a.) r = vakio ⇒ pallokuori

Huom. Jos r = 0 ⇒ piste

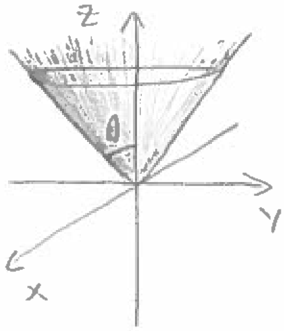


b.) ϕ = vakio ⇒ puoli-taso



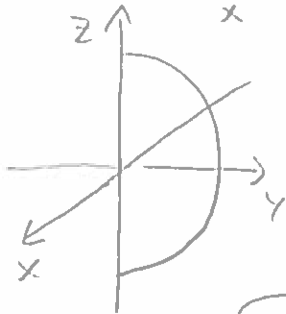
c.) θ = vakio ⇒ ympyräkartio

Huom. Jos $\theta = \frac{\pi}{2}$ ⇒ taso



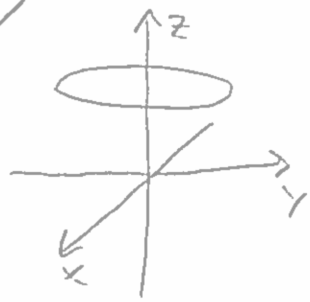
d.) r = vakio ja ϕ = vakio ⇒ puoliympyrän kehä

Huom. Jos r = 0 ⇒ piste

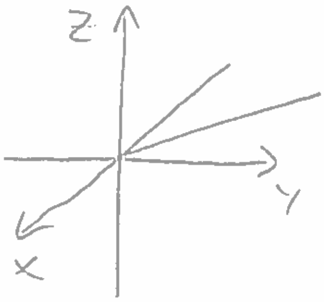


e.) r = vakio ja θ = vakio ⇒ ympyrän kehä

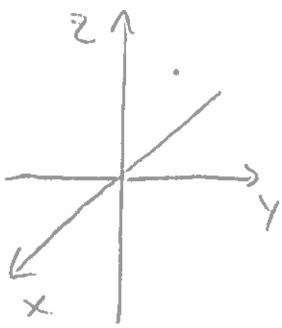
Huom. Jos r = 0 ⇒ piste



f.) ϕ = vakio ja θ = vakio ⇒ puolisuora



g.) r = vakio, ϕ = vakio ja θ = vakio ⇒ piste



FYSA114 Demo5

$\boxed{2.}$ Lasketaan sylinterikoordinaatiston yksikkövektorit:

$$\vec{r}(\rho,\phi,z) = \rho\cos\phi\,\hat{\imath} + \rho\sin\phi\,\hat{\jmath} + z\,\hat{k}$$

$$\frac{\partial\vec{r}}{\partial\rho} = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}, \quad \left|\frac{\partial\vec{r}}{\partial\rho}\right| = 1 = h_\rho \Rightarrow \hat{\rho} = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}$$

$$\frac{\partial\vec{r}}{\partial\phi} = -\rho\sin\phi\,\hat{\imath} + \rho\cos\phi\,\hat{\jmath}, \quad \left|\frac{\partial\vec{r}}{\partial\phi}\right| = \rho = h_\phi \Rightarrow \hat{\phi} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}$$

$$\frac{\partial\vec{r}}{\partial z} = \hat{k} \quad \left|\frac{\partial\vec{r}}{\partial z}\right| = 1 = h_z \Rightarrow \hat{z} = \hat{k}$$

Lasketaan pallokoordinaatiston yksikkövektorit:

$$\vec{r}(r,\phi,\theta) = r\sin\theta\cos\phi\,\hat{\imath} + r\sin\theta\sin\phi\,\hat{\jmath} + r\cos\theta\,\hat{k}$$

$$\frac{\partial\vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k} = \hat{r}$$

$$\left|\frac{\partial\vec{r}}{\partial r}\right| = (\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta)^{1/2} = 1 = h_r$$

$$\frac{\partial\vec{r}}{\partial\phi} = -r\sin\theta\sin\phi\,\hat{\imath} + r\sin\theta\cos\phi\,\hat{\jmath}$$

$$\left|\frac{\partial\vec{r}}{\partial\phi}\right| = (r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi)^{1/2} = r\sin\theta = h_\phi$$

$$\Rightarrow \hat{\phi} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}$$

$$\frac{\partial\vec{r}}{\partial\theta} = r\cos\theta\cos\phi\,\hat{\imath} + r\cos\theta\sin\phi\,\hat{\jmath} - r\sin\theta\,\hat{k}$$

$$\left|\frac{\partial\vec{r}}{\partial\theta}\right| = (r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta)^{1/2} = r = h_\theta$$

$$\Rightarrow \hat{\theta} = \cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k}$$

Piste $(x,y,z) = (1,1,1)$

Sylinterikoord.

$(r,\phi,z) = (\sqrt{2}, \frac{\pi}{4}, 1)$

$\hat{\rho} = \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$

$\hat{\phi} = -\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$

$\hat{z} = \hat{k}$

Pallokoord.

$(r,\phi,\theta) = (\sqrt{3}, \frac{\pi}{4}, \tan^{-1}\sqrt{2})$

$\hat{r} = \frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}\hat{\imath} + \frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}\hat{\jmath} + \frac{1}{\sqrt{3}}\hat{k} = \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k})$

$\hat{\phi} = \frac{1}{\sqrt{2}}(-\hat{\imath} + \hat{\jmath})$

$\hat{\theta} = \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}\hat{\jmath} - \frac{\sqrt{2}}{\sqrt{3}}\hat{k} = \frac{1}{\sqrt{6}}(\hat{\imath} + \hat{\jmath} - 2\hat{k})$

FYSA114 Demo5

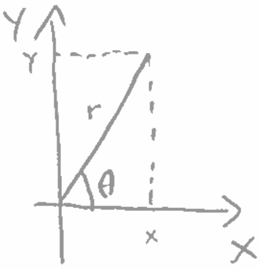
$x = r\cos\theta$
$y = r\sin\theta$

$\boxed{3.}$ $u(r,\theta)$, $\vec{V} = V_r(r,\theta)\,\hat{r} + V_\theta(r,\theta)\,\hat{\theta}$

$$\nabla u = (\hat{\imath}\,\partial_x + \hat{\jmath}\,\partial_y)u = \hat{\imath}\left(\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}\right) + \hat{\jmath}\left(\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}\right)$$

$$= \frac{\partial u}{\partial r}\underbrace{\left(\hat{\imath}\,\frac{\partial r}{\partial x} + \hat{\jmath}\,\frac{\partial r}{\partial y}\right)}_{\frac{x}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{y}{\sqrt{x^2+y^2}}\hat{\jmath} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} = \hat{r}} + \frac{\partial u}{\partial \theta}\left(\hat{\imath}\,\frac{\partial \theta}{\partial x} + \hat{\jmath}\,\frac{\partial \theta}{\partial y}\right)$$

$r = \sqrt{x^2 + y^2}$
$\theta = \arctan\left(\frac{y}{x}\right)$



$$\frac{\partial \theta}{\partial x} = \frac{1}{\left(\frac{y}{x}\right)^2 + 1}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{y^2 + x^2} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{\left(\frac{y}{x}\right)^2 + 1}\cdot\frac{1}{x} = \frac{x}{y^2 + x^2} = \frac{x}{r^2} = \frac{\cos\theta}{r}$$

$$\Rightarrow \hat{\imath}\,\frac{\partial \theta}{\partial x} + \hat{\jmath}\,\frac{\partial \theta}{\partial y} = \frac{1}{r}\underbrace{\left(-\frac{y}{r}\hat{\imath} + \frac{x}{r}\hat{\jmath}\right)}_{\hat{\theta}} = \frac{1}{r}\hat{\theta}$$

$$\Rightarrow \nabla u = \frac{\partial u}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\theta}$$

Entä $\nabla\cdot\vec{V}_r$?

$$\nabla = \hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} = \hat{\imath}\left(\frac{\partial r}{\partial x}\cdot\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\right) + \hat{\jmath}\left(\frac{\partial r}{\partial y}\cdot\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\cdot\frac{\partial}{\partial \theta}\right)$$

$$\vec{V} = V_r\,\hat{r} + V_\theta\,\hat{\theta} = V_r(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) + V_\theta(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})$$
$$= (V_r\cos\theta - V_\theta\sin\theta)\hat{\imath} + (V_r\sin\theta + V_\theta\cos\theta)\hat{\jmath}$$

$$\Rightarrow \nabla\cdot\vec{V} = \underbrace{\left(\frac{\partial r}{\partial x}\cdot\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\right)}_{x\text{-derivaatta}}\underbrace{\left(V_r\cos\theta - V_\theta\sin\theta\right)}_{x\text{-komponentti}} + \overbrace{\left(\frac{\partial r}{\partial y}\cdot\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\cdot\frac{\partial}{\partial \theta}\right)}^{y\text{-derivaatta}}\cdot$$

y-komponentti.

$$(V_r\sin\theta + V_\theta\cos\theta)$$

$$= \frac{\partial r}{\partial x}\left(\cos\theta\,\frac{\partial V_r}{\partial r} - \sin\theta\,\frac{\partial V_\theta}{\partial r}\right) + \frac{\partial \theta}{\partial x}\left(\frac{\partial V_r}{\partial \theta}\cos\theta - \sin\theta\,V_r - \frac{\partial V_\theta}{\partial \theta}\sin\theta - V_\theta\cos\theta\right)$$
$$+ \frac{\partial r}{\partial y}\left(\sin\theta\,\frac{\partial V_r}{\partial r} + \cos\theta\,\frac{\partial V_\theta}{\partial r}\right) + \frac{\partial \theta}{\partial y}\left(\frac{\partial V_r}{\partial \theta}\sin\theta + V_r\cos\theta + \frac{\partial V_\theta}{\partial \theta}\cos\theta - V_\theta\sin\theta\right)$$

FYSA114 Demo 5

$\boxed{3.}$

$$= \frac{\partial V_r}{\partial r}\left(\frac{\partial r}{\partial x}\cos\theta + \frac{\partial r}{\partial y}\sin\theta\right) + \frac{\partial V_\theta}{\partial r}\left(-\frac{\partial r}{\partial x}\sin\theta + \frac{\partial r}{\partial y}\cos\theta\right)$$

$$+ \frac{\partial V_r}{\partial \theta}\left(\frac{\partial \theta}{\partial x}\cos\theta + \frac{\partial \theta}{\partial y}\sin\theta\right) + \frac{\partial V_\theta}{\partial \theta}\left(-\frac{\partial \theta}{\partial x}\sin\theta + \frac{\partial \theta}{\partial y}\cos\theta\right)$$

$$+ V_r\left(-\frac{\partial \theta}{\partial x}\sin\theta + \frac{\partial \theta}{\partial y}\cos\theta\right) + V_\theta\left(-\frac{\partial \theta}{\partial x}\cos\theta - \frac{\partial \theta}{\partial y}\sin\theta\right)$$

$$= \frac{\partial V_r}{\partial r}\left(\cos^2\theta + \sin^2\theta\right) + \frac{\partial V_\theta}{\partial r}\left(-\cos\theta\sin\theta + \sin\theta\cos\theta\right)$$

$$+ \frac{\partial V_r}{\partial \theta}\left(-\frac{\sin\theta}{r}\cos\theta + \frac{\cos\theta}{r}\sin\theta\right) + \frac{\partial V_\theta}{\partial \theta}\left(\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right)$$

$$+ V_r\left(+\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right) + V_\theta\left(\frac{\sin\theta\cos\theta}{r} - \frac{\cos\theta\sin\theta}{r}\right)$$

$$= \frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{1}{r}V_r = \frac{1}{r}\cdot r\frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial r}{\partial r}V_r + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}$$

$$= \frac{1}{r}\frac{\partial (rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}$$

FYSA114 Demo5

4. Näytä, että sylinterikoordinaatistossa:

$$\nabla^2 u(r,\phi,z) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}, \quad \text{kun}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$r = \sqrt{x^2+y^2}$

$\phi = \arctan\left(\frac{y}{x}\right)$

$x = r\cos\phi$

$y = r\sin\phi$

1° $\frac{\partial^2 u}{\partial z^2}$ pysyy sellaisenaan.

2° $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x}$, $\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$ $\Leftarrow$ Ketjusääntö

$$\frac{\partial u}{\partial x} = \left(\frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} - \frac{1}{\left(\frac{y}{x}\right)^2+1}\cdot\frac{y}{x^2}\frac{\partial}{\partial \phi}\right)u = \left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\right)u$$

$$= \cos\phi\frac{\partial u}{\partial r} - \frac{\sin\phi}{r}\frac{\partial u}{\partial \phi}$$

$$\frac{\partial^2 u}{\partial x^2} = \left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\right)\left(\cos\phi\frac{\partial u}{\partial r} - \frac{\sin\phi}{r}\frac{\partial u}{\partial \phi}\right)$$

$$= \cos^2\phi\frac{\partial^2 u}{\partial r^2} - \cos\phi\sin\phi\left(-\frac{1}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r}\frac{\partial^2 u}{\partial r\partial \phi}\right)$$
$$- \frac{\sin\phi}{r}\left(-\sin\phi\frac{\partial u}{\partial r} + \cos\phi\frac{\partial^2 u}{\partial \phi\partial r}\right) + \frac{\sin\phi}{r^2}\left(\cos\phi\frac{\partial u}{\partial \phi} + \sin\phi\frac{\partial^2 u}{\partial \phi^2}\right)$$

$$= \cos^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{\cos\phi\sin\phi}{r^2}\frac{\partial u}{\partial \phi} - \frac{\cos\phi\sin\phi}{r}\frac{\partial^2 u}{\partial r\partial \phi}$$
$$+ \frac{\sin^2\phi}{r}\frac{\partial u}{\partial r} - \frac{\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial \phi\partial r} + \frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{\sin^2\phi}{r^2}\frac{\partial^2 u}{\partial \phi^2}$$

$$= \cos^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{2\cos\phi\sin\phi}{r^2}\frac{\partial u}{\partial \phi} - \frac{2\cos\phi\sin\phi}{r}\frac{\partial^2 u}{\partial r\partial \phi}$$
$$+ \frac{\sin^2\phi}{r^2}\frac{\partial^2 u}{\partial \phi^2}$$

FYSA114 Demo 5

$\boxed{4.}$ 3° $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial u}{\partial y}$, $\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$

$$\frac{\partial u}{\partial y} = \left(\frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} + \frac{1}{\left(\frac{y}{x}\right)^2+1}\cdot\frac{1}{x}\frac{\partial}{\partial \phi}\right)u = \left(\frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} + \frac{x}{y^2+x^2}\frac{\partial}{\partial \phi}\right)u$$

$$= \left(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}\right)u = \sin\phi\frac{\partial u}{\partial r} + \frac{\cos\phi}{r}\frac{\partial u}{\partial \phi}$$

$$\frac{\partial^2 u}{\partial y^2} = \left(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}\right)\left(\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\phi}{r}\frac{\partial u}{\partial \phi}\right)$$

$$= \sin^2\phi\frac{\partial^2 u}{\partial r^2} + \sin\phi\cos\phi\left(-\frac{1}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r}\frac{\partial^2 u}{\partial r\partial \phi}\right)$$
$$+ \frac{\cos\phi}{r}\left(\cos\phi\frac{\partial u}{\partial r} + \sin\phi\frac{\partial^2 u}{\partial \phi\partial r}\right) + \frac{\cos\phi}{r^2}\left(-\sin\phi\frac{\partial u}{\partial \phi} + \cos\phi\frac{\partial^2 u}{\partial \phi^2}\right)$$

$$= \sin^2\phi\frac{\partial^2 u}{\partial r^2} - \frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial \phi} + \frac{\cos^2\phi}{r}\frac{\partial u}{\partial r}$$
$$+ \frac{\cos\phi\sin\phi}{r}\frac{\partial^2 u}{\partial \phi\partial r} - \frac{\cos\phi\sin\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{\cos^2\phi}{r^2}\frac{\partial^2 u}{\partial \phi^2}$$

$$= \sin^2\phi\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial \phi} + \frac{\cos^2\phi}{r}\frac{\partial u}{\partial r}$$
$$+ \frac{\cos^2\phi}{r^2}\frac{\partial^2 u}{\partial \phi^2}$$

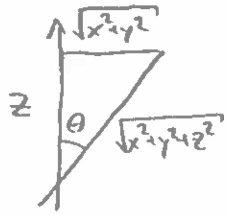
$$\Rightarrow \nabla^2 u = \cos^2\phi\frac{\partial^2 u}{\partial r^2} + \sin^2\frac{\partial^2 u}{\partial r^2} + \frac{\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\phi}{r}\frac{\partial u}{\partial r}$$
$$+ \frac{\sin^2\phi}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cos^2\phi}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}$$

Jei!

FYSA114 Demo 5

$\boxed{5.}$ Olk. $\vec{F} = F_r\hat{r} + F_\theta\hat{\theta} + F_\phi\hat{\phi}$. Näytä, että



$$\nabla\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} = \hat{\imath}\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\right)$$

$$+\hat{\jmath}\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\right) + \hat{k}\left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}\right)$$

$$r = (x^2+y^2+z^2)^{1/2}$$

$$\theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\phi = \arctan\left(\frac{y}{x}\right)$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \cos\phi\sin\theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \sin\phi\sin\theta$$

$$\frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \cos\theta$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{\frac{x^2+y^2}{z^2}+1}\cdot\frac{1}{z}\cdot\frac{x}{\sqrt{x^2+y^2}} = \frac{z}{x^2+y^2+z^2}\cdot\frac{x}{\sqrt{x^2+y^2}} = \frac{\cos\theta\cos\phi}{r}$$

$$\frac{\partial\theta}{\partial y} = \frac{z}{x^2+y^2+z^2}\cdot\frac{y}{\sqrt{x^2+y^2}} = \frac{\cos\theta\sin\phi}{r}$$

$$\frac{\partial\theta}{\partial z} = \frac{1}{\frac{x^2+y^2}{z^2}+1}\cdot\left(-\frac{\sqrt{x^2+y^2}}{z^2}\right) = -\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial\phi}{\partial x} = \frac{1}{\left(\frac{y}{x}\right)^2+1}\left(-\frac{y}{x^2}\right) = -\frac{y}{y^2+x^2} = -\frac{\sin\phi}{\sqrt{y^2+x^2}} = -\frac{\sin\phi}{\sqrt{r^2\sin^2\phi\sin^2\theta + r^2\cos^2\phi\sin^2\theta}}$$

$$= -\frac{\sin\phi}{r\sin\theta}$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{\left(\frac{y}{x}\right)^2+1}\cdot\frac{1}{x} = \frac{x}{y^2+x^2} = \frac{\cos\phi}{r\sin\theta}, \quad \frac{\partial\phi}{\partial z} = 0.$$

FYSA114 Demo5

$\boxed{5.}$ Esitetään $\vec{F}$ vektorien $\hat{\imath}, \hat{\jmath}$ ja $\hat{k}$ avulla.

$$\vec{F} = F_r \underbrace{(\sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k})}_{\hat{r}\ \text{(ks. teht. 2)}} + F_\theta \underbrace{(\cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k})}_{\hat{\theta}}$$

$$+ F_\phi \underbrace{(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})}_{\hat{\phi}}$$

$$= (F_r\sin\theta\cos\phi + F_\theta\cos\theta\cos\phi - F_\phi\sin\phi)\hat{\imath}$$

$$+ (F_r\sin\theta\sin\phi + F_\theta\cos\theta\sin\phi + F_\phi\cos\phi)\hat{\jmath}$$

$$+ (F_r\cos\theta - F_\theta\sin\theta)\hat{k}$$

$$= F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$

$$\frac{\partial F_x}{\partial x} = \left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\right)F_x$$

$$= \left(\cos\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot$$

$$(F_r\sin\theta\cos\phi + F_\theta\cos\theta\cos\phi - F_\phi\sin\phi)$$

$$= \cos^2\phi\sin^2\theta\frac{\partial F_r}{\partial r} + \cos^2\phi\sin\theta\cos\theta\frac{\partial F_\theta}{\partial r} - \cos\phi\sin\theta\sin\phi\frac{\partial F_\phi}{\partial r}$$

$$+ \frac{\cos\theta\cos^2\phi}{r}\left(\frac{\partial F_r}{\partial\theta}\sin\theta + F_r\cos\theta\right) + \frac{\cos\theta\cos^2\phi}{r}\left(\frac{\partial F_\theta}{\partial\theta}\cos\theta - F_\theta\sin\theta\right)$$

$$- \frac{\cos\theta\cos\phi\sin\phi}{r}\frac{\partial F_\phi}{\partial\theta}$$

$$- \frac{\sin\phi}{r}\left(\frac{\partial F_r}{\partial\phi}\cos\phi - F_r\sin\phi\right) - \frac{\sin\phi\cos\theta}{r\sin\theta}\left(\frac{\partial F_\theta}{\partial\phi}\cos\phi - F_\theta\sin\phi\right)$$

$$+ \frac{\sin\phi}{r\sin\theta}\left(\frac{\partial F_\phi}{\partial\phi}\sin\phi + F_\phi\cos\phi\right)$$

FYSA114 Demo5

$\boxed{5.}$

$$\frac{\partial F_y}{\partial y} = \left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}\right)F_y$$

$$= \left(\sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot$$

$$\left(F_r\sin\theta\sin\phi + F_\theta\cos\theta\sin\phi + F_\phi\cos\phi\right)$$

$$= \sin^2\phi\sin^2\theta\frac{\partial F_r}{\partial r} + \sin^2\phi\sin\theta\cos\theta\frac{\partial F_\theta}{\partial r} + \sin\theta\sin\phi\cos\phi\frac{\partial F_\phi}{\partial r}$$

$$+ \frac{\cos\theta\sin^2\phi}{r}\left(\frac{\partial F_r}{\partial\theta}\sin\theta + F_r\cos\theta\right) + \frac{\cos\theta\sin^2\phi}{r}\left(\frac{\partial F_\theta}{\partial\theta}\cos\theta - F_\theta\sin\theta\right)$$

$$+ \frac{\cos\theta\sin\phi\cos\phi}{r}\frac{\partial F_\phi}{\partial\theta}$$

$$+ \frac{\cos\phi}{r}\left(\frac{\partial F_r}{\partial\phi}\sin\phi + F_r\cos\phi\right) + \frac{\cos\phi\cos\theta}{r\sin\theta}\left(\frac{\partial F_\theta}{\partial\phi}\sin\phi + F_\theta\cos\phi\right)$$

$$+ \frac{\cos\phi}{r\sin\theta}\left(\frac{\partial F_\phi}{\partial\phi}\cos\phi - F_\phi\sin\phi\right)$$

$$\frac{\partial F_z}{\partial z} = \left(\frac{\partial r}{\partial z}\cdot\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}\right)F_z$$

$$= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\cdot\left(F_r\cos\theta - F_\theta\sin\theta\right)$$

$$= \cos^2\theta\frac{\partial F_r}{\partial r} - \cos\theta\sin\theta\frac{\partial F_\theta}{\partial r} - \frac{\sin\theta}{r}\left(\frac{\partial F_r}{\partial\theta}\cos\theta - F_r\sin\theta\right)$$

$$+ \frac{\sin\theta}{r}\left(\frac{\partial F_\theta}{\partial\theta}\sin\theta + F_\theta\cos\theta\right)$$

Kasataan sitten tulokset.

FYSA114 Demo 5

$\boxed{5.}$

$$\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{\partial F_r}{\partial r}\left(\cos^2\phi \sin^2\theta + \sin^2\phi \sin^2\theta + \cos^2\theta\right) \qquad \text{ok}$$

$$+ \frac{\partial F_\theta}{\partial r}\left(\cos^2\phi \sin\theta\cos\theta + \sin^2\phi \sin\theta\cos\theta - \cos\theta\sin\theta\right) \qquad \text{ok } (=0)$$

$$+ \frac{\partial F_\phi}{\partial r}\left(-\cos\phi\sin\theta\sin\phi + \sin\theta\sin\phi\cos\phi\right) \qquad \text{ok } (=0)$$

$$+ \frac{\partial F_r}{\partial \theta}\left(\frac{\cos\theta\sin\theta\cos^2\phi}{r} + \frac{\cos\theta\sin\theta\sin^2\phi}{r} - \frac{\sin\theta\cos\theta}{r}\right) \qquad \text{ok } (=0)$$

$$+ F_r\left(\frac{\cos^2\theta\cos^2\phi}{r} + \frac{\cos^2\theta\sin^2\phi}{r} + \frac{\sin^2\theta}{r}\right) \qquad \text{ok}$$

$$+ \frac{\partial F_\theta}{\partial \theta}\left(\frac{\cos^2\theta\cos^2\phi}{r} + \frac{\cos^2\theta\sin^2\phi}{r} + \frac{\sin^2\theta}{r}\right) \qquad \text{ok}$$

$$+ F_\theta\left(-\frac{\cos\theta\sin\theta\cos^2\phi}{r} - \frac{\cos\theta\sin\theta\sin^2\phi}{r} + \frac{\sin\theta\cos\theta}{r}\right) \qquad \text{ok } (=0)$$

$$+ \frac{\partial F_\phi}{\partial \theta}\left(-\frac{\cos\theta\cos\phi\sin\phi}{r} + \frac{\cos\theta\sin\phi\cos\phi}{r}\right) \qquad \text{ok } (=0)$$

$$+ \frac{\partial F_r}{\partial \phi}\left(-\frac{\sin\phi\cos\phi}{r} + \frac{\cos\phi\sin\phi}{r}\right) \qquad \text{ok } (=0)$$

$$+ F_r\left(\frac{\sin^2\phi}{r} + \frac{\cos^2\phi}{r}\right) \qquad \text{ok}$$

$$+ \frac{\partial F_\theta}{\partial \phi}\left(-\frac{\sin\phi\cos\theta\cos\phi}{r\sin\theta} + \frac{\cos\phi\cos\theta\sin\phi}{r\sin\theta}\right) \qquad \text{ok } (=0)$$

$$+ F_\theta\left(\frac{\sin^2\phi\cos\theta}{r\sin\theta} + \frac{\cos^2\phi\cos\theta}{r\sin\theta}\right) \qquad \text{ok}$$

$$+ \frac{\partial F_\phi}{\partial \phi}\left(\frac{\sin^2\phi}{r\sin\theta} + \frac{\cos^2\phi}{r\sin\theta}\right) \qquad \text{ok}$$

$$+ F_\phi\left(\frac{\sin\phi\cos\phi}{r\sin\theta} - \frac{\cos\phi\sin\phi}{r\sin\theta}\right) \qquad \text{ok } (=0)$$

FYSA114 Demo 5

$\boxed{5.}$

$$= \frac{\partial F_r}{\partial r} + \frac{1}{r} F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{1}{r} F_r + \frac{\cos\theta}{r\sin\theta} F_\theta + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

$$= \frac{\partial F_r}{\partial r} + \frac{2}{r} F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\cos\theta}{r\sin\theta} F_\theta + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

$$= \frac{1}{r^2}\left(r^2 \frac{\partial F_r}{\partial r} + 2r F_r\right) + \frac{1}{r\sin\theta}\left(\frac{\partial F_\theta}{\partial \theta}\sin\theta + F_\theta \cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(F_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

Hiphei ∇!

Vinkkejä pitkiin tehtäviin:

1. Jaa ongelma osiin, esim. laske termit $\frac{\partial F_x}{\partial x}$ kukin erikseen ja yhdistä saamasi tulokset lopuksi

2. Kirjoita selkeitä kokonaisuuksia esim. yksi asia yhdelle riville. Näin voit jäljittää virheet helpommin.

3. Kirjoita väljästi ja jätä hiukan tilaa korjauksille.