VA tentti 9.5.2014

① $f(x,y,z) = z e^{x^2-y^2}$, $\vec{u} = 2\hat{i} + 2\hat{j} + \hat{k}$

a) Suunnatun derivaatan pisteessä (x,y,z) on

$$\frac{\partial f}{\partial u} = \nabla f \cdot \hat{u}.$$

Nyt

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$
$$= (2xz\,\hat{i} - 2yz\,\hat{j} + \hat{k})e^{x^2-y^2}$$

Pisteessä $(1,-1,2)$ $\nabla f = 4\hat{i} + 4\hat{j} + \hat{k}$. Niinpä

$$\frac{\partial f}{\partial u}\Big|_{(1,-1,2)} = (4\hat{i} + 4\hat{j} + \hat{k}) \cdot \frac{2\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{2^2+2^2+1}}$$
$$= \frac{1}{3}(4\cdot 2 + 4\cdot 2 + 1)$$
$$= \frac{17}{3}.$$

3p

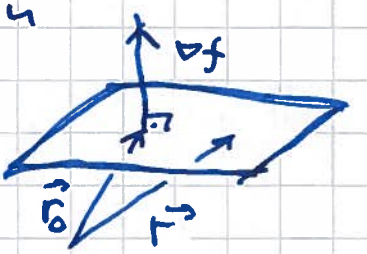
b) f kasvaa nopeiten, eli $\frac{\partial f}{\partial u}$ on suurin, silloin kun

$$\hat{u} \uparrow\uparrow \nabla f$$

eli suunnassa $\nabla f\big|_{(1,-1,2)} = 4\hat{i} + 4\hat{j} + \hat{k}$.

2p

c) Pinta $f(x,y,z) = 2$. Pisteeseen $\vec{r}_0 = (1,-1,2)$ piirretyn tangenttitason yhtälö on



$$(\vec{r} - \vec{r}_0) \cdot \nabla f = 0$$

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (4\hat{i} + 4\hat{j} + \hat{k})$$
$$= (\hat{i} - \hat{j} + 2\hat{k}) \cdot (4\hat{i} + 4\hat{j} + \hat{k})$$

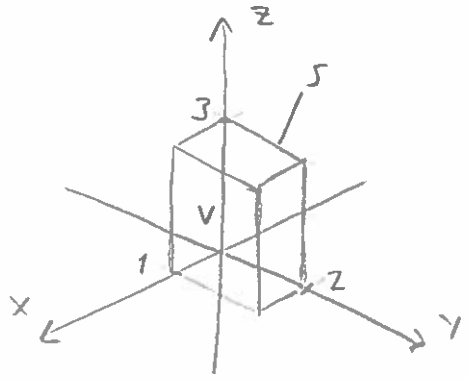
$$4x + 4y + z = 4 - 4 + 2 = 2.$$

3p

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2.

$$\vec{F} = x^2 y\,\hat{i} - 2yz\,\hat{j} + x^3 y^2\,\hat{k}$$
$$\vec{G} = y^2 z\,\hat{i} - 2zx\,\hat{j} + x^3 y^2\,\hat{k}$$



2

Alue on suljettu.
⇒ Käytetään Gaussia.

$\nabla \cdot \vec{F} = 2xy - 2z$ 1

$\nabla \cdot \vec{G} = 0$ 1

2

$$\oint_S \vec{F}\cdot d\vec{S} \overset{\text{Gauss}}{=} \int_V \nabla\cdot\vec{F}\,dV = \int_0^1 dx \int_0^2 dy \int_0^3 dz\,(2xy - 2z)$$

$$= \int_0^1 dx \int_0^2 dy \Big/_0^3 2xyz - z^2 = \int_0^1 dx \int_0^2 dy\,(6xy - 9)$$

$$= \int_0^1 dx \Big/_0^2 3xy^2 - 9y = \int_0^1 dx\,(12x - 18) = \Big/_0^1 6x^2 - 18x = 6 - 18$$

$= -12$ 1

$$\oint_S \vec{G}\cdot d\vec{S} \overset{\text{Gauss}}{=} \int_V \nabla\cdot\vec{G}\,dV = \int_V 0\,dV = 0$$ 1

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3 $\vec{u} = (x^2+y^2)\hat{i} + \sqrt{1+y^3}\,\hat{j}$

a.) $\nabla \cdot \vec{u} = 2x + 3y^2 \cdot \frac{1}{2}(1+y^3)^{-1/2}$

$= 2x + \frac{3y^2}{2\sqrt{1+y^3}}$ 2

$$\nabla \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2+y^2 & \sqrt{1+y^3} & 0 \end{vmatrix}$$

$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-2y)$

$= -2y\hat{k}$ 2

b.) Laske $\int_C \vec{u} \cdot d\vec{r}$ kun $C: \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $t \in [0, \pi]$

$\vec{r}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$

$$\int_C \vec{u} \cdot d\vec{r} = \int_0^\pi [\underbrace{(\cos^2 t + \sin^2 t)}_{=1}\hat{i} + \sqrt{1+\sin^3 t}\,\hat{j}] \cdot (-\sin t\,\hat{i} + \cos t\,\hat{j})\,dt$$

$$= \int_0^\pi (-\sin t + \underbrace{\cos t \sqrt{1+\sin^3 t}}_{\text{vaikea integraali :'}})\,dt$$

$\Rightarrow$ Kokeillaan Stokesilla.

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3. Täydennetään polku puoliympyräksi.



Alkuperäinen integraali saadaan kun suljetun loopin integraalista vähennetään integraali C_1:in yli.

Suljettu looppi:

Stokes $\Rightarrow$ $\oint_{C,C_1} \vec{u}\cdot d\vec{r} = \int_A \nabla\times\vec{u}\cdot d\vec{s}$ 1

Nyt $d\vec{s} = \hat{k}\,ds$

$$\int_A (-2y\hat{k})\cdot\hat{k}\,ds = -2\int_A y\,ds = -2\int_0^1 dr \int_0^{\pi} d\theta\, r\sin\theta$$

$$= -2\int_0^1 dr\, r^2 \underbrace{\Big/_0^{\pi} -\cos\theta}_{=1+1} = -4\int_0^1 dr\, r^2 = -4\Big/_0^1 \frac{1}{3}r^3 = -\frac{4}{3}$$ 1

C_1: $\vec{r}(t) = (t-1)\hat{i}$, $t\in[0,2]$

$\vec{r}\,'(t) = \hat{i}$

$$\Rightarrow \int_{C_1}\vec{u}\cdot d\vec{r} = \int_0^2 [((t-1)^2+0^2)\hat{i} + \sqrt{1+0^3}\,\hat{j}]\cdot\hat{i}\,dt$$ 1

$$= \int_0^2 (t-1)^2\,dt = \Big/_0^2 \frac{1}{3}(t-1)^3 = \frac{1}{3}(2-1)^3 - \frac{1}{3}(0-1)^3 = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$$

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3. Nyt saadaan lopullinen tulos:

$$\int_C \vec{u}\cdot d\vec{r} = \oint_{C,C_1} \vec{u}\cdot d\vec{r} - \int_{C_1} \vec{u}\cdot d\vec{r} \qquad 1$$

$$= -\frac{4}{3} - \frac{2}{3} = \underline{\underline{-2}}$$

Tapa 2

$$\int_0^{\pi} \left(-\sin t + \cos t\sqrt{1+\sin^3 t}\right) dt \qquad 1$$

$$= \Big|_0^{\pi} \cos t + \int_0^{\pi} \cos t\sqrt{1+\sin^3 t}\, dt \qquad \text{mv. } z = t - \frac{\pi}{2}$$

$$= -2 + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(z+\frac{\pi}{2}\right)\sqrt{1+\sin^3\left(z+\frac{\pi}{2}\right)}\, dz$$

1

$$= -2 + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{(-\sin z)}_{\text{pariton}} \underbrace{\sqrt{1+\cos^3 z}}_{\text{parillinen}}\, dz \qquad 1$$

pariton

$= 0$ koska symmetriset rajat.

$$= -2 \qquad 1$$

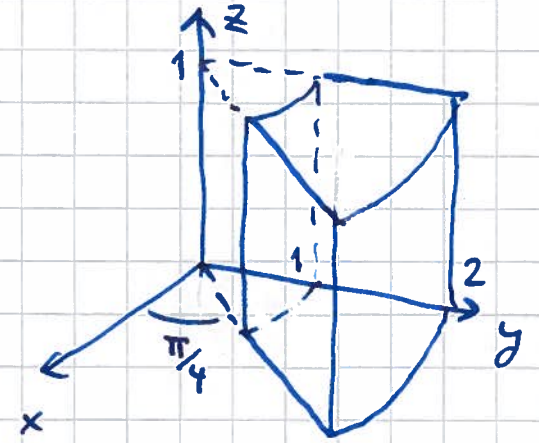
(4) $f(x,y) = x^2+y^2$

a) Alue $1 \le x^2+y^2 \le 4$, $0 \le x \le y$, $0 \le z \le 1$

Sylinteri koordinaatisto:

$\varrho = \sqrt{x^2+y^2}$; $f = \varrho^2$

$\begin{cases} x = \varrho\cos\theta \\ y = \varrho\sin\theta \\ z \end{cases}$



Alue:

$1 \le \varrho \le 2$, $0 \le \varrho\cos\theta \le \varrho\sin\theta$ | : ϱ

$0 \le \cos\theta \le \sin\theta$

$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$ ← $\cos\theta \ge 0$

↑ $\cos\theta \le \sin\theta$

$0 \le z \le 1$

Silloin (4p)

$$\iiint_D f\,dV = \int_{\pi/4}^{\pi/2}\int_0^1\int_1^2 \varrho^2\cdot\varrho\,d\theta\,dz\,d\varrho$$

$$= \int_{\pi/4}^{\pi/2} d\theta \int_0^1 dz \int_1^2 \varrho^3\,d\varrho = \frac{\pi}{4}\cdot 1\cdot\left[\frac{\varrho^4}{4}\right]_1^2 = \frac{\pi}{16}(16-1) = \frac{15}{16}\pi.$$

b) Alue $x^2+y^2+z^2 \le a^2$, a-säteinen pallo.

Pallokoordinaatisto: $0 \le r \le a$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$

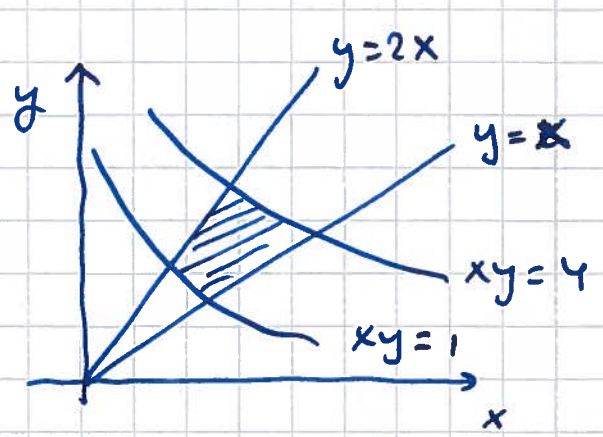
$f = x^2+y^2 = (r\sin\theta\cos\phi)^2 + (r\sin\theta\sin\phi)^2 = (r\sin\theta)^2$

$$\iiint_D f\,dV = \int_0^a dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\,(r\sin\theta)^2 \underbrace{r^2\sin\theta\,dr\,d\theta\,d\phi}_{dV}$$ (4p)

$$= \int_0^a r^4\,dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3\theta\,d\theta = \frac{a^5}{5}\cdot 2\pi\cdot\frac{4}{3} = \frac{8\pi}{15}a^5.$$

$$\int_0^{\pi}\sin\theta\,d\theta - \int_0^{\pi}\sin\theta\cos^2\theta\,d\theta = \left[-\cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{\pi} = \frac{4}{3}$$

(5) a) Ks. luennot (3p)



b) Muuttujien vaihto:

$$\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases}$$

Tällöin alue on uv-tasossa suorakaide

$$\begin{cases} 1 \le u \le 4 \\ 1 \le v \le 2. \end{cases}$$

Muunnoksen Jacobin determinantti on

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{vmatrix} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix}$$

$$= \frac{y}{x} - \left(-\frac{y}{x}\right) = 2\frac{y}{x} = 2v$$

$$\rightarrow \quad du\,dv = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dx\,dy,$$

joten

$$dA = dx\,dy = \left| \frac{\partial(u,v)}{\partial(x,y)} \right|^{-1} du\,dv = \frac{1}{2v}\,du\,dv$$

Pinta-ala: $A = \int dA = \int_{u=1}^{4} \int_{v=1}^{2} \frac{1}{2v}\,du\,dv = \int_1^4 du \int_1^2 \frac{dv}{2v}$

$$= [u]_1^4 \left[\tfrac{1}{2}\ln v\right]_1^2 = 3 \cdot \tfrac{1}{2}\ln 2 = \tfrac{3}{2}\ln 2$$

(3p)

$$I = \iint \underbrace{y^2}_{uv}\,dx\,dy = \iint uv\,\frac{1}{2v}\,du\,dv = \frac{1}{2}\int_1^2 dv \int_1^4 u\,du$$

$$= \frac{1}{2}[v]_1^2 \left[\tfrac{1}{2}u^2\right]_1^4 = \frac{1}{2} \cdot 1 \cdot \frac{1}{2}(16-1) = \frac{15}{4}$$

(2p)

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6. Osoita:

$$\oint_C \underbrace{(y\,dx + z\,dy + x\,dz)}_{\vec{F}\cdot d\vec{r}} = \sqrt{3}\pi a^2$$

Kun C on pintojen $\underbrace{x^2+y^2+z^2=a^2}_{a\text{-säteinen pallo}}$ ja $\underbrace{x+y+z=0}_{\text{taso origon läpi.}}$ leikkauskäyrä.

$\Rightarrow$ C on a-säteinen ympyrä (eli siis myös suljettu) 2

Nyt $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} = \hat{i}(0-1) - \hat{j}(1-0) + \hat{k}(0-1)$$

$$= -\hat{i} - \hat{j} - \hat{k} \quad 1$$

C:n rajaaman ympyrän sisäosa on tason $x+y+z=0$ suuntainen. Tämän tason normaalivektorit ovat $\hat{N} = \pm\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$. 1

$$\oint_C \vec{F}\cdot d\vec{r} \overset{\text{Stokes}}{=} \int \nabla\times\vec{F}\cdot d\vec{s} = \mp\int (\hat{i}+\hat{j}+\hat{k})\cdot\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})\,ds$$

1 (C:n sisäosa) (C:n sisäosa) 1

$$= \mp\sqrt{3}\int_{C\text{:n sisäosa}} ds = \mp\sqrt{3}\pi a^2$$

1 1

Kun normaaliksi valitaan $-\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$ saadaan haluttu tulos.