Matematiikan peruskurssi
Exercise 5
23.2.2017

1. With what values of a and b does the system of linear equations

$$\begin{cases} 2x_1 & & -2x_3 & = 10 \\ & 4x_2 & +4x_3 & = 8 \\ -2x_1 & +4x_2 & +ax_3 & = b \end{cases}$$

have zero/one/infinite number of solutions?

2. In previous exercises we computed the determinant of the coefficient matrix of the system of equations

$$\begin{cases} -4x_1 + 2x_2 & & + 4x_4 & = 10 \\ 2x_1 & & +4x_3 + 4x_4 & = 0 \\ 6x_1 - 2x_2 & & +2x_3 - 2x_4 & = 2 \\ 9x_1 - 2x_2 & & +2x_3 + x_4 & = 2 \end{cases}$$

and it was 24. This means that there exists a unique solution (x_1, x_2, x_3, x_4). Find x_3 using Cramer's rule.

3. Sketch the graph of function $f : [-1, 2] \to \mathbb{R}$, $f(x) = x^2 - 1$. Define M_f and A_f, and also find them in your sketch.

4. What is the range of the function $f : [-1, 1] \to \mathbb{R}$, $f(x) = x^2 - 2x + 1$? Justify your answer.

5. Find the domain M_f and range A_f of the function $f(x) = \frac{1}{x^2}$, and justify your answer.

6. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 1$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = e^x + x^2$. Find the expressions of the functions $f \circ f$, $f \circ g$, and $g \circ f$.

7. Investigate if the following functions are injections, surjections, bijections:

$$\begin{aligned} &f : \mathbb{R} \to [-2, \infty), && f(x) = x^2 + 2x - 1, \\ &g : [0, \infty) \to \mathbb{R}, && g(x) = x^2 - 8, \\ &h : [-1, 1] \to [-1, 3], && h(x) = 3 - (x + 1)^2 \end{aligned}$$

8. Define inverse functions for the functions in exercise 7 (for those functions that you can invert). Also, sketch the graph(s) of the inverse function(s), and locate in your sketch the domains and ranges of the original function and of the inverse function.

9. Define the inverse function of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1 - 2x$, i.e. f^{-1}, and draw in the same picture the graphs of both f and f^{-1}.

(10*). Define the inverse function of $f : [-1, 1] \to [-1, 1]$,

$$f(x) = \begin{cases} x, \text{ jos } x \leq 0, \\ x^2, \text{ jos } x > 0, \end{cases}$$

and justify your answer.[1]
(Hint: Since the function f is defined in pieces, probably the function f^{-1} should also be defined in pieces)

(11*). Assume that $f : A \to \mathbb{R}$ is strictly increasing, that is for all $x, z \in A$ the condition $x < z$ implies $f(x) < f(z)$. Prove that f is an injection. Then $f : A \to A_f$ is a bijection, so that there is an inverse function $f^{-1} : A_f \to A$. Is the function f^{-1} strictly increasing?

(12*). (Linear regression) In statistics, one is interested in the system of linear equations $AX = B$, where A is $m \times n$-matrix, X is $n \times 1$-matrix, and B is $m \times 1$-matrix, **and furthermore,** $m > n$. Since $m > n$, usually the system of equations has no solution. Still it is possible to find a matrix X, for which AX is "as close as possible"to the matrix B. This means, that the sum $\sum_{i=1}^{m} |(AX)_i - B_i|^2$ is minimized (this is the "least squares"method). When $m = n$ and A is invertible, the solution is $X = A^{-1}B$. If $m > n$, and $A^T A$ is invertible, the minimization-problem has a unique solution given by $X = (A^T A)^{-1} A^T B$. Prove that this indeed holds, at least in the case when

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

[1] Guess the expression of the inverse function, and prove that this really is the inverse function of f.