Matematiikan peruskurssi
Exercises 7
9.3.2017

In the exercises below, the phrase "find the local extremes"means, that find the local maxima and minima, and find the values of the function at these points.

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1. Draw the graph of a function $f : [-2, 2] \to \mathbb{R}$ that satisfies all of the below:

- f is continuous,
- $f'(x) < 0$ when $x \in (-2, 0)$,
- $f(0) = 1$,
- $f'(x) > 0$ when $x \in (0, 1)$,
- $f(1) = 2$,
- $f'(x) = 0$, when $x \in (1, 2)$.

(Hint: You don't need to find an expression for the function, just draw a suitable graph)

2. Find the local extremes of the function $f : [-2, 2] \to \mathbb{R}$, $f(x) = x^2 - 2x + 3$. Are some of these also global extremes?

3. Find the local extremes of the function $f : [0, 4] \to \mathbb{R}$, $f(x) = -x^3 + 6x^2 - 12x + 8$. Are some of these also global extremes?

4. Find the local extremes of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4 - 2x^2$.

5. Find the local extremes of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x^2+1}$.

6. Approximate the number $\sqrt[3]{997} = 997^{\frac{1}{3}}$ using the tangent line drawn to the graph of the function $f(x) = \sqrt[3]{x}$ at point $(1000, f(1000))$.

7. Compute the indefinite integrals

$$(a) \int x^3 dx, \quad (b) \int -3\sqrt{x} dx, \quad (c) \int \frac{x+1}{x^2} dx.$$

8. Compute the indefinite integrals

$$(a) \int e^{2x} - e^{-x} dx, \quad (b) \int 3xe^{x^2} dx, \quad (c) \int \frac{2x+1}{x^2+x+5} dx.$$

9. Find, for the function $f(x) = \sin(x) - e^{-\frac{x}{2}}$, an antiderivative F such that $F(0) = -1$.

10. Compute the indefinite integral $\int \frac{2}{x^2-1} dx$.
(Hint: First find $c, d \in \mathbb{R}$ such that $\frac{2}{x^2-1} = \frac{c}{x-1} + \frac{d}{x+1}$)

(11*). There is an outbreak of Ebola at a certain district (not Jyväskylä). Number of people diseased with Ebola at time x is approximately described by the function $E : [0,1] \to [0,\infty)$, $E(x) = 9481.5 \cdot (x^3 - x^4)$. When is the number of diseased people the largest, and how many of them are there at that time?

(12*). Approximate the number $\cos(\frac{\pi}{3} + \frac{1}{\sqrt{3}})$ using the tangent line drawn to the graph of the function $f(x) = \cos(x)$ at point $(\frac{\pi}{3}, f(\frac{\pi}{3}))$.

(13*). Assume that f is differentiable at x_0, and assume that f has a local minimum (or a local maximum) at x_0. Prove that $f'(x_0) = 0$.
(Hint: Investigate the sign of the difference quotient $\frac{f(x_0+h)-f(x_0)}{h}$ when $h > 0$ and when $h < 0$)

(14*). Prove **Rolle's theorem:** Assume that f is continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b), and furthermore, assume that $f(a) = 0 = f(b)$. Prove that there exists $c \in (a, b)$ such that $f'(c) = 0$.
(Hint: A continuous function attains it's minimum and maximum on a closed interval. Try to apply exercise (13*))

(15*). Prove the **mean value theorem:** Assume that f is continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b). Prove that there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
Hint: Apply Rolle's theorem to the function

$$g(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b-a}(x-a) \right]$$

(16*). Assume that $f : (a, b) \to \mathbb{R}$ is continuous, and let $F : (a, b) \to \mathbb{R}$ and $G : (a, b) \to \mathbb{R}$ be antiderivatives of the function f. Investigate the monotonicity of the function $F - G$, and deduce what kind of (elementary) function it must be.

(17*). Find the local extremes of the function $f : [0, \frac{\pi}{2}] \to \mathbb{R}$, $f(x) = x + 2\cos x$.

(18*). Find the local extremes of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x^2 - 1|$.

(19*). Find the domain and the local extremes of the function $f(x) = -\frac{3}{2}\log(x^2 + 2) + x$.