Matematiikan peruskurssi
Exercise 8
16.3.2017

1. Compute the indefinite integral $\int x^2 e^x \mathrm{d}x$ using integration by parts (twice).

2. Compute the indefinite integrals $\int \frac{4x^3+8x}{x^4+4x^2+6}\mathrm{d}x$ and $\int \frac{2x}{(x^2+1)^2}\mathrm{d}x$.
(Hint: you can use the substitutions $t = x^4 + 4x^2 + 6$ ja $t = x^2 + 1$)

3. Compute the indefinite integral $\int 2x^3 e^{x^2}\mathrm{d}x$.
(Hint: integration by parts, choose $f'(x) = 2xe^{x^2}$ and $g(x) = x^2$)

4. Compute the indefinite integral $\int 3x^2 \log(x^3)\mathrm{d}x$ using the substitution $t = x^3$.

5. Compute the indefinite integral $\int e^{x^{\frac{1}{3}}}\mathrm{d}x$ using the substitution $x = g(t) = t^3$.

6. Calculate $\int_0^1 f(x)\mathrm{d}x$ when

$$(a)\ f(x) = 3x^2 + 2x - 2, \quad (b)\ f(x) = e^{-x}, \quad (c)\ f(x) = \sin(1 - x).$$

7. Calculate $\int_1^2 x\log(x)\mathrm{d}x$.

8. Calculate $\int_3^8 \frac{\cos(\sqrt{x+1})}{\sqrt{x+1}}\mathrm{d}x$ using the substitution $t = \sqrt{x+1}$.

9. Calculate $\int_1^4 \sqrt{\sqrt{x} - 1}\mathrm{d}x$.
(Hint: use for example the substitution $x = (t^2 + 1)^2$ or $x = (t + 1)^2$)

10. Calculate the improper integrals

$$(a) \int_1^\infty \frac{1}{x^3}\mathrm{d}x, \quad (b) \int_0^1 \frac{1}{x^{\frac{1}{3}}}\mathrm{d}x.$$

(11*). Compute the indefinite integral $\int \log(x)^2\mathrm{d}x$.
(Hint: check in the lectures how we computed $\int \log(x)dx$)

(12*). Calculate

$$(a) \int_0^{\frac{\pi}{2}} \sin(x)\cos^3(x)\mathrm{d}x, \quad (b) \int_0^{\frac{\pi}{2}} \sin^3(x)\cos^3(x)\mathrm{d}x.$$

(Hint: Pythagorean theorem may be useful in part (b))

(13*). Compute the indefinite integral $\int \frac{1}{x\log(x)}\mathrm{d}x$.
(Hint: use the substitution $t = \log(x)$)

(14*) Calculate

(a) $\frac{d}{dx}\int_0^x e^{t^2}\mathrm{d}t$.
(Hint: DON'T try to integrate, instead use the fundamental theorem of calculus. The task is to differentiate the function $F(x) = \int_0^x e^{t^2}dt$ with respect to variable x)

(b) $\frac{d}{dx}\int_0^{-2x} e^{t^2}\mathrm{d}t$.
(Hint: Chain rule and part (a))

(15*). Let $f : [a, b] \to \mathbb{R}$ be continuous, and let M be the largest value of the function f on the interval $[a, b]$, and let m be the smallest value of the function f on the interval $[a, b]$. Prove that

$$m(b-a) \leq \int_a^b f(x)\mathrm{d}x \leq M(b-a).$$

(Hint: Monotonicity of the integral)

(16*). Prove the **mean value theorem of integration**: Assume that $f : [a, b] \to \mathbb{R}$ is continuous. Prove that there exists $c \in [a, b]$ such that

$$f(c)(b-a) = \int_a^b f(x)\mathrm{d}x.$$

(Hint: Set $k = \frac{\int_a^b f(x)\mathrm{d}x}{b-a}$, and deduce using Bolzano's theorem and exercise (15*), that function f must attain value k at some point between $[a, b]$)

(17*). Assume that $f : [a, b] \to \mathbb{R}$ is continuous. Prove that for all $x \in (a, b)$ it holds

$$\lim_{h\to 0}\frac{1}{2h}\int_{x-h}^{x+h} f(t)dt = f(x).$$

(Hint: Use the mean value theorem of integration, and the continuity of function f)