Matematiikan peruskurssi
Exercise 9
23.3.2017

1. Calculate the volume of the area that is between the graphs of the functions $f(x) = x$ and $g(x) = x^2$ on the interval $[0, 1]$, and also sketch the area.

2. Calculate the volume of the area that is between the graphs of the functions $f(x) = x^3$ and $g(x) = -x$ on the interval $[-1, 1]$, and also sketch the area.

3. Connect the differential equations with corresponding verbal descriptions:

$$\begin{array}{llllll} (a) & y' = 2y, & (b) & y' = -(y-2), & (c) & y' = \dfrac{2}{y}, \\ (d) & y' = 2x^2 + 2, & (e) & y' = 2y^2, & (f) & y' = -2xy, \end{array}$$

(1) The rate of change of the function is inversely proportional to the value of the function

(2) The rate of change of the function is directly proportional to the value of the function

(3) The rate of change of the function is a polynomial of the variable

(4) The rate of change of the function is directly proportional to the product of the function and the variable

(5) The rate of change of the function is directly proportional to the square of the function

(6) The rate of change of the function is directly proportional to the difference of the function and number 2

4. For the solutions of the differential equations in exercise 3, compute the value of the derivative when $x = 1$ and $y = y(1) = 3$. Is this solution function increasing or decreasing at point $x = 1$?

5. Solve the differential equations

$$(a)\ y'(x) = 2x - 7, \quad (b)\ y'(x) = x\ln(x), \quad (c)\ y'(x) = 3x^2 e^{x^3}.$$

6. Solve the initial value problem

$$\begin{cases} y'(x) = \ln(x) + 1 \\ y(1) = 6 \end{cases}$$

7. Prove that $y(x) = \frac{1-e^x}{1+e^x}$ is a solution to the differential equation $y'(x) = \frac{(y(x)^2-1)}{2}$.

8. Solve the differential equation

$$y'(x) = (x + x^2)(1 + 2x)e^{x+x^2}$$

9. Solve the separable differential equation

$$y'(x) = \frac{1}{y(x)}.$$

10. Solve the separable differential equation

$$y'(x) = 3x^2 y(x).$$

(11*). Ask something (related to the course, and in written please!). (Hint: is there something in the course that remains unclear?)

(12*). Solve the initial value problem

$$\begin{cases} y'(x) = \frac{1}{y(x)}, \\ y(0) = -2. \end{cases}$$

What is the interval of solution of this problem?

(13*). A company has investment possibilities A and B. During a year, A produces the cashflow (negative number means that the investment produces loss) according to the function[1] $f : [0,4] \to \mathbb{R}$, $f(t) = -32t^3 + 108t^2 - 20t - 10$ and B according to function $g : [0,4] \to \mathbb{R}$, $g(t) = 8t^3 + 15t^2 - 30t - 100$. Here t is time in quarters of years.

(a) Which investment cumulates more profit during the year (i.e. between $[0,4]$)?

(b) The CEO is only interested in making as much money as possible during the first quarter of the year. Which investment cumulates more profit during this time period (i.e. between $[0,1]$)?

[1] So at time t the "amount of profit" of A equals $f(t)$.

(14*). Solve the differential equations

$$(a)\ y'(x) = \sin x + \frac{1}{x}, \quad (b)\ y'(x) = x^2 e^x + \frac{1}{2\sqrt{x}}, \quad (c)\ y'(x) = \log(x).$$

(15*). You go fish. During the first hour you travel to the lake, set your gear, and find a good place to catch fish. After this you set to action: function

$$f : [0,\infty) \to [0,\infty), \quad f(t) = \begin{cases} 0, \ \text{kun } t < 1, \\ \frac{1}{2} t^{-\frac{3}{2}}, \ \text{kun } t \geq 1 \end{cases},$$

describes the instantaneous probability of catching a fish at time t (hours).[2] What is the probability that you do not catch a fish during the first 4 hours? How about during the first 16 hours?

(16*). Your friend is also fishing. He lives near the lake, so when you started to fish, he had already been there for an hour: function

$$f : [0,\infty) \to [0,\infty), \quad f(t) = \frac{\log(2)}{4} \cdot e^{-\frac{\log(2)}{4} t},$$

describes the instantaneous probability of catching a fish at time t (hours).[3] What is the probability that your friend does not catch a fish during the first 4 hours? How about during the first 16 hours?

(17*). Functions $y(x) = Ce^x$ are solutions to the differential equation

$$y'(x) - y(x) = 0 \tag{1}$$

for any $C \in \mathbb{R}$. Prove that there are no other solutions.
(Hint: Assume that $y_0 : \mathbb{R} \to \mathbb{R}$, $y_0(x)$, is a solution to (1). Differentiate the function $y_0(x)e^{-x}$, and see what happens.)

[2] This is a Pareto-distribution.
[3] This is an exponential distribution.