QUANTUM MECHANICS I A (FYSA231), spring 2010

Exercise 3.

1. A system consisting of two identical non-interacting particles has the following Hamilton's operator (k=1,2)

$$\hat{H} = \hat{H}_1 + \hat{H}_2, \quad \hat{H}_k = -\frac{\hbar^2}{2m}\nabla_k^2 + V(\vec{r}_k).$$

The one particle eigenvalue equation is

$$\hat{H}_k \phi_{n_k}(\vec{r}_k) = E_{n_k}\phi_{n_k}(\vec{r}_k).$$

Show that the eigenstates of $\hat{H}$ has the following form

$$\phi_{n_1 n_2}(\vec{r}_1, \vec{r}_2) = \phi_{n_1}(\vec{r}_1)\phi_{n_2}(\vec{r}_2),$$

and the eigen values

$$E_{n_1 n_2} = E_{n_1} + E_{n_2}.$$

2. Schrödinger's equation of one dimensional harmonic oscillator is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi(x) = E\psi(x)$$

Make scaling change $x = by$, where $b^2 = \frac{\hbar}{m\omega}$ and use notation $\epsilon = \frac{2E}{\hbar\omega}$ and modify the differential equation into following form

$$\frac{d^2\phi}{dx^2} + (\epsilon - y^2)\phi = 0,$$

where $\psi(x) = \psi(by) = \phi(y)$.
Continues. Use

$$\phi = f(y)e^{-\frac{1}{2}y^2},$$

and insert it into result 1 and show that you get

$$f'' - 2f'y + (\epsilon - 1)f = 0,$$

which is so-called Hermite's differential equation.

3. Continues. Use Frobenius' series method and insert

$$f(y) = y^s \sum_{n=0}^{\infty} a_n y^n$$

into the differential equation and show that you get the following recurrence relation.

$$a_{n+2}(n + s + 2)(n + s + 1) = a_n(2n + 2s - \epsilon + 1)$$

4. a) When you were trying to get the recurrence you noticed that $s = 0$ and $a_1 = 0$ is one possibility. Insert these into the recurrence formula and show that with large n following relation is true.

$$\frac{a_{n+2}}{a_n} \approx \frac{2}{n}.$$

   Show that e^{y^2} behaves in the same way. You see that when you use Taylor's series for e^{y^2}. From this one can conclude that f(y) and $\phi(y)$ are diverging when $y \to \infty$.

   b) The recurrence formula must brake so that the wave function is normalizabe. Find the expression for the energy levels by using this information.

5. a) Write down the ground state wave function and the energy of the quantum mechanical harmonic oscillator.

   b) Potential is now changing rapidly so that wave function remains the same but the frequency ω is doubled to 2ω. One measures the energy immediately. Show that one can not obtain the old ground state energy anymore.

   c) In which probability one gets $\hbar\omega$ in the new system?