QUANTUM MECHANICS I A (FYSA231), spring 2010

Exercise 4.

1. $\mathbf{U}$ is arbitrary unitary matrix.
   a) Show that $\mathbf{U}\mathbf{U}^\dagger = \mathbf{1}$.
   b) Show that, if $\mathbf{A}' = \mathbf{U}\mathbf{A}\mathbf{U}^\dagger$, then $\mathbf{A} = \mathbf{U}^\dagger\mathbf{A}'\mathbf{U}$.
   c) Show that $(\mathbf{A}\mathbf{B})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$.
   d) Show that, if $\mathbf{A}$ is Hermitian then $\mathbf{A}' = \mathbf{U}\mathbf{A}\mathbf{U}^\dagger$ is also Hermitian
2. $\hat{H}$ is Hamilton's operator and $\hat{A}$ is another operator corresponding to some observable. $\{\mid 1\rangle, \mid 2\rangle, \mid 3\ \rangle\ \}$ is orthonormalized base of the system. Define the matrix representations for operators $\hat{H}$ and $\hat{A}$ when following relation are true.
$$\begin{cases} \hat{H} \mid 1\rangle = \hbar\omega \mid 1\rangle \\ \hat{H} \mid 2\rangle = 2\hbar\omega \mid 2\rangle \\ \hat{H} \mid 3\rangle = 3\hbar\omega \mid 3\rangle \end{cases} \qquad \begin{cases} \hat{A} \mid 1\rangle = \lambda \mid 2\rangle \\ \hat{A} \mid 2\rangle = \lambda \mid 1\rangle \\ \hat{A} \mid 3\rangle = 2\lambda \mid 3\rangle \end{cases}$$
   Find the normalized eigenvectors and eigenvalues of $\hat{A}$. What is the $\hat{H}$ in the base of $\hat{A}$?
3. One dimensional harmonic oscillator is in the ground state. The energy and the normalized wave function are
$$E_0 = \frac{1}{2}\hbar\omega,\ \psi_0 = (\pi b^2)^{-1/4} e^{(\frac{-x^2}{2b^2})}.$$
   missä $b^2 = \hbar/(m\omega)$.
   a) Calculate the expectation values for potential energy and for the kinetic energy.
   b) Calculate $\Delta x \Delta p$ and comment the result remembering the uncertainty principle.
   c) Calculate the most probable value for x.
4. $\phi_1(\vec{r})$ and $\phi_2(\vec{r})$ are ortonormalized states of Hamilton's operator. Observable A do not have explicit time dependence, $A \neq A(t)$. System is in the normalized state described by
$$\psi(t, \vec{r}) = c_1\phi_1(\vec{r})e^{-iE_1t/\hbar} + c_2\phi_2(\vec{r})e^{-iE_2t/\hbar}.$$
   Use notations
$$\hbar\omega = E_1 - E_2,\quad A_{mn} = \int \phi_m^* \hat{A}\phi_n d^3r.$$
   and define the expectation value $\langle A\rangle_t$. Show that this value oscillates between two values with period
$$T = \frac{2\pi\hbar}{|E_1 - E_2|}.$$

5. a) Show that $[\hat{x}^n, \hat{p}] = i\hbar n x^{n-1}$

b) Show by using one particle Hamiltonian and

$$\frac{d}{dt}\langle A\rangle = \langle\frac{\partial A}{\partial t}\rangle + \frac{i}{\hbar}\langle[\hat{H}, \hat{A}]\rangle,$$

that

$$\frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m},$$

and

$$\frac{d\langle p\rangle}{dt} = -\langle\frac{dV(x)}{dx}\rangle.$$

Hint: $V(\hat{x}) = \sum_{n=0}^{\infty} a_n \hat{x}^n$