Bayes Teoria, tehtävät-4, ratkaisut

1. Jos $y_1, \ldots, y_n$ ovat ehdollisesti riipumattomia ehdolla θ, $p(y|\theta) \sim \mathcal{N}(\theta, v)$ ja $\theta \sim \mathcal{N}(m, w)$ jossa tunnetaan v, w, m silloin a posteriori $\theta|y_1, \ldots, y_n \sim \mathcal{N}(m_1, w_1)$

$$w_1 = \left(\frac{1}{w} + \frac{n}{v}\right)^{-1} = \frac{w}{k} \text{ jos ja vain jos}$$
$$k = 1 + n\frac{w}{v}$$

Huomaat muuttuneesta notaatiosta.

2.

$$p(\theta|\text{ data}) \sim \exp\bigl(-\theta^2/18 + \theta 2/9\bigr)$$

täydentämällä neliön saadaan

$$= \exp\left(-\frac{1}{2}\frac{(\theta-2)^2}{9}\right)\exp\left(-\frac{2^2}{9}\right)$$

siis $p(\theta|\text{ data})$ on gaussinen $\mathcal{N}(2, 9)$.

Sen 90%-HDI on $(2 - 1.644854 \times \sqrt{9}, 2 + 1.644854 \times \sqrt{9}) = (-2.934561, 6.934561)$

3. Olkoon $(y_1, \ldots, y_n)$ ovat ehdollisesti riippumattomia ja $\mathcal{N}(\theta, 1)$ jakautunet ehdolla θ. $\pi(\theta) \propto \mathbf{1}(\theta) > 0$. Tämä priori on epäaito, ei ole todennösköisyysmitta.

Kuitenkin posteriori sattaa olla aito todennäköisyysmitta.

$$p(\theta|y_1, \ldots, n) \propto \mathbf{1}(\theta > 0)\exp\bigl(\theta\sum_{i=1}^{n} y_i - \frac{n}{2}\theta^2\bigr)$$
$$\propto \mathbf{1}(\theta > 0)\exp\left(-\frac{1}{2/n}\bigl(\theta - \overline{y}_n\bigr)^2\right)$$
$$\text{jossa } \overline{y}_n = n^{-1}\sum_{i=1}^{n} y_i$$

Kun $n > 0$, posteriori on ehdollistettu gaussinen jakauma

$$p(\theta|y_1, \ldots, n) = \frac{\mathbf{1}(\theta > 0)}{1 - \Phi(-\sqrt{n}\overline{y}_n)}\sqrt{\left(\frac{n}{2\pi}\right)}\exp\left(-\frac{1}{2/n}\bigl(\theta - \overline{y}_n\bigr)^2\right)$$

jossa $\Phi(t) = P(G \le t)$ on standardi normali gaussisen kertymäfunktio.

$$1 - \Phi(-\sqrt{n}\overline{y}_n) = \Phi(\sqrt{n}\overline{y}_n)$$

4. Olkoon $\varphi = \frac{\theta}{1-\theta}$, $\theta = \frac{\varphi}{1+\varphi}$.

$$\int \mathbf{1}_A(\varphi(\theta))p_\theta(\theta)d\theta = \int \mathbf{1}_A(\varphi)p_\theta(\theta(\varphi)\frac{d\theta(\varphi)}{d\varphi}d\varphi$$
$$= \int \mathbf{1}_A(\varphi)p_\varphi(\varphi)d\varphi$$
$$p_\varphi(\varphi) = p_\theta(\theta(\varphi))\frac{d\theta(\varphi)}{d\varphi}$$

Koska

$$\frac{d\theta(\varphi)}{d\varphi} = \frac{1}{(1+\varphi)} - \frac{\varphi}{(1+\varphi)^2} = \frac{1}{(1+\varphi)^2}$$

seuraa

$$p_\varphi(\varphi) = \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+s)\Gamma(\beta+n-s)}\left(\frac{\varphi}{1+\varphi}\right)^{\alpha+s-1}\left(\frac{1}{1+\varphi}\right)^{\beta+n-s-1}\frac{1}{(1+\varphi)^2}$$
$$= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+s)\Gamma(\beta+n-s)}\varphi^{(\alpha+s-1)}\left(\frac{1}{1+\varphi}\right)^{-(\alpha+\beta+n)}$$

5.

$$P(X_{n+1} = 1|X_1 = 1, \ldots, X_n = 1) = \frac{\alpha+n}{\alpha+\beta+n}$$

jossa a priori $\theta \sim \mathrm{Beta}(\alpha, \beta)$. Arvot $\alpha = \beta 1$ vastaavat tasaista jakaumaa välissä $[0, 1]$. Kun θ a priori on $[0, 1]$-tasaisesti jakautunut, seuraa

$$\frac{\alpha+n}{\alpha+\beta+n} = \frac{n+1}{n+2}$$

6.

$$E(y_{n+1}|y_1, \ldots, y_n) = \int_{\mathbb{R}} E(y_{n+1}|\theta)p(\theta|y_1, \ldots, y_n)d\theta$$
$$= \int_{\mathbb{R}} \theta p(\theta|y_1, \ldots, y_n)d\theta = m_1$$

$\mathrm{Var}(y_{n+1}|y_1, \ldots, y_n) = E(\mathrm{Var}(y_{n+1}|\theta)|y_1, \ldots, y_n) + \mathrm{Var}(E(y_{n+1}|\theta)|y_1, \ldots, y_n) = E(1|y_1, \ldots, y_n) + \mathrm{Var}(\theta|y_1, \ldots, y_n) = 1 + w_1$

7. $y_1, \ldots, y_n|\theta$ ehdollisesti riippumattomia jakaumalla $\sim$ Tasainen$(0, \theta)$. $\pi(\theta) \propto \frac{1}{\theta}$, $\theta > 0$.

$$p(\theta|y_1, \ldots, y_n) \propto \mathbf{1}(\max_i y_i \leq \theta)\theta^{-(n+1)}$$

$$\theta^{-(n+1)} = -\frac{1}{n}\frac{d}{d\theta}\theta^{-n}$$

$$\int_{\max y_i}^{\infty} \theta^{-(n+1)} d\theta = \frac{1}{n}(\max y_i)^{-n}$$

Siis

$$p(\theta|y_1, \ldots, y_n) = n\mathbf{1}(\max_i y_i \leq \theta)(\max y_i)^n \theta^{-(n+1)}$$

Kun $n = 6$ ja $\max y_i = 1.96$

$$P(\theta > 2.5|y_1, \ldots, y_n) = n(1.96)^n \int_{\max\{2.5, 1.96\}}^{\infty} \theta^{-(n+1)} d\theta$$

$$= \left(\frac{1.96}{2.5}\right)^n = 0.2322183$$