Bayes Teoria, ratkaisut-3, (28.01.2011)

1.

$$
\begin{aligned}
&p(\theta_1|Y=0) = \frac{p(Y=0|\theta_1)p(\theta_1)}{p(Y=0|\theta_1)p(\theta_1)+p(Y=0|\theta_2)p(\theta_2)}\\
&= \frac{5/6\times 1/2}{5/6\times 1/2 + 2/3\times 1/2} = 5/9\\
&p(\theta_2|Y=0) = 1-p(\theta_1|Y=0) = 4/9\\
&p(\theta_1|Y=1) = \frac{p(Y=1|\theta_1)p(\theta_1)}{p(Y=1|\theta_1)p(\theta_1)+p(Y=1|\theta_2)p(\theta_2)}\\
&= \frac{1/6\times 1/2}{1/6\times 1/2 + 1/3\times 1/2} = 1/3\\
&p(\theta_2|Y=1) = 1-p(\theta_1|Y=1) = 2/3
\end{aligned}
$$

$$
\begin{aligned}
\frac{p(\theta_1)}{1-p(\theta_1)} &= \frac{p(\theta_1)}{p(\theta_2)} = \frac{1/2}{1/2} = 1\\
\frac{p(\theta_1|Y=1)}{1-p(\theta_1|Y=1)} &= \frac{p(\theta_1|Y=1)}{p(\theta_2|Y=1)} = \frac{1/3}{2/3} = 1/2\\
\frac{p(\theta_1|Y=0)}{1-p(\theta_1|Y=0)} &= \frac{p(\theta_1|Y=0)}{p(\theta_2|Y=0)} = \frac{5/9}{4/9} = 5/4
\end{aligned}
$$

Kun priori muutetaan $p(\theta_1) = 1 - p(\theta_2) = 5/6$

$$
\frac{p(\theta_1)}{1-p(\theta_1)} = \frac{p(\theta_1)}{p(\theta_2)} = \frac{5/6}{1/6} = 5
$$

$$
\begin{aligned}
&p(\theta_1|Y=0) = \frac{5/6\times 5/6}{5/6\times 5/6 + 2/3\times 1/6} = 25/29\\
&p(\theta_2|Y=0) = 1-p(\theta_1|Y=0) = 4/29\\
&p(\theta_1|Y=1) = \frac{1/6\times 5/6}{1/6\times 5/6 + 1/3\times 1/6} = 5/7\\
&p(\theta_2|Y=1) = 1-p(\theta_1|Y=1) = 2/7\\
&\frac{p(\theta_1|Y=0)}{p(\theta_2|Y=0)} = 25/4\\
&\frac{p(\theta_1|Y=1)}{p(\theta_2|Y=1)} = 5/2
\end{aligned}
$$

2.

$$
p(x|\theta) = \begin{cases} 2\theta x \exp(-\theta x^2) & \text{kun } x > 0 \\ 0 & \text{muuten} \end{cases}
$$

$$F(x|\theta) = P(X \le x|\theta) = \int_0^x 2\theta t \exp(-\theta t^2) dt$$

$$-\int_0^x \frac{d}{dt} \exp(-\theta t^2) dt = \Big[\exp(-\theta t^2)\Big]_x^0 = 1 - \exp(-\theta x^2)$$

$$S(x|\theta) = P(X > x|\theta) = 1 - F(x|\theta) = \exp(-\theta x^2)$$

$$h(x|\theta) = -\frac{d}{dx} \log S(x|\theta) = 2\theta x$$

$$= \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\big(X \in [t + \Delta t] \big| X \ge t\big)$$

3.

$$p(x|\theta) = \begin{cases} \theta \exp(-\theta x) & \text{kun } x > 0 \\ 0 & \text{muuten} \end{cases}$$

$$P(X > x|\theta) = \int_x^\infty \theta \exp(-\theta t) dt = \exp(-\theta x)$$

$$P(\sqrt{X} > x|\theta) = P(X > x^2|\theta) = \exp(-\theta x^2)$$

4. Olkoon posteriori jakauma

$$p(\theta|data = y) = \frac{1}{2}\theta^2 \exp(-\theta), \quad \theta > 0$$

Tämä on gamma jakauma parametreilla $(3, 1)$.

$$E(\theta^n|y) = \frac{1}{2} \int_0^\infty \theta^{n+2} \exp(-\theta) d\theta =$$

$$\frac{\Gamma(3+n)}{\Gamma(3)} = \frac{(2+n)!}{2!}$$

jossa kun $\alpha > 0$

$$\Gamma(\alpha) = \int_0^\infty \theta^{\alpha-1} \exp(-\theta) d\theta$$

toteuttaa $\Gamma(1) = 1$, $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ ja siksi $\Gamma(n) = (n-1)!$. Siksi $E(\theta|y) = 3$, $E(\theta^2|y) = 12$, $\mathrm{Var}(\theta|y) = 12 - 3^2 = 3$, siis keskihajonta on $\sqrt{3}$

Voidaan myös laskea jakauman kertymäfunktio. Osittaisintegroinnilla

$$F(t|y) = P(\theta \le t|y) = \frac{1}{2} \int_0^t \theta^2 \exp(-\theta) d\theta =$$

$$-\frac{1}{2}\exp(-t)t^2 + \int_0^t \theta \exp(-\theta) d\theta = -\frac{1}{2}\exp(-t)t^2 - \exp(-t)t + \int_0^t \exp(-\theta) d\theta =$$

$$-\frac{1}{2}\exp(-t)t^2 - \exp(-t)t + 1 - \exp(-t) = 1 - \exp(-t)(1 + t + t^2/2)$$

90%-symmetrinen luottamus väli on (a, b) jolla $F(a|y) = 0.05$ ja $1 - F(b|y) = 0.05$, Saadaan epälineaarinen yhtälö

$$\exp(-t)(1 + t + t^2/2) = \alpha$$

joka ratkaistaan numerisesti.

5. 90% HDI: etsitään väli (a,b) jolla

$$p(a|y) = p(b|y) \quad \text{ja } F(b|y) - F(a|y) = \alpha$$

jossa $\alpha = 0.9$. Siis ratkaistaan epälineaarinen yhtälö

$$\begin{aligned}
&a^2 \exp(-a) = b^2 \exp(-b) \\
&\exp(-a)(1 + a + a^2/2) - \exp(-b)(1 + b + b^2/2) = \alpha
\end{aligned}$$

6. Kun priori on $p(\theta_1) = p(\theta_2) = 1/2$

$$\begin{aligned}
&p(\widetilde{Y} = 1|Y = 1) = p(\widetilde{Y} = 1,\ \theta_1|Y = 1) + p(\widetilde{Y} = 1,\ \theta_2|Y = 1) = \\
&p(\widetilde{Y} = 1|\theta_1, Y = 1)p(\theta_1|Y = 1) + p(\widetilde{Y} = 1|\theta_2, Y = 1)p(\theta_2|Y = 1) \\
&= p(\widetilde{Y} = 1|\theta_1)p(\theta_1|Y = 1) + p(\widetilde{Y} = 1|\theta_2)p(\theta_2|Y = 1) \\
&= 1/6 \times 1/3 + 1/3 \times 2/3 = 5/18
\end{aligned}$$

Kun priori $p(\theta_1) = 1 - p(\theta_2) = 5/6$

$$p(\widetilde{Y} = 1|Y = 1) = 1/6 \times 5/7 + 1/3 \times 2/7 = 3/14$$

7. (Cavendish data)

$$\begin{aligned}
&\theta \sim \mathcal{N}\Big(m, w) \quad (\text{ priori }) \\
&y_i|\theta \sim \mathcal{N}\Big(\theta, v) \quad 1 \leq i \leq n \\
&(\text{ ehdollisesti riippumattomia ja samoin jakautuneita }), \\
&\theta|y_1, \ldots, y_n \sim \mathcal{N}\Big(m_1, w_1) \quad (\text{ posteriori }) \\
&w_1 = \Big(1/w + n/v)^{-1}, \quad m_1 = w_1\Big(m/w + n\bar{y}_n/v\Big) \\
&\bar{y}_n = \frac{1}{n}\sum_{i=1}^{n} y_i = 5.48 \quad n = 23, \\
&w_1 = \big(10000/4 + 23 \times 100/4\big)^{-1} = 4/12300 = 3.252 \times 10^{-4} \\
&m_1 = 3.252 \times 10^{-4}\big(5.48 \times 10000/4 + 23 \times 5.48 \times 100/4\big) = \\
&\frac{4}{12300}\,\frac{5.48 \times 12300}{4} = 5.48
\end{aligned}$$

8. Winbugs koodi:

```
model {
theta ~ dnorm( m,w)
for(i in 1: n) {
y[i] ~ dnorm( theta,v )
}
}

data
list( n=23, m=5.48, w=0.0004,v=0.04,
y=c( 5.36,5.29,5.58,5.65,5.57,5.53,5.62,5.29,5.44,5.34,
5.79,5.1,5.27,5.39,5.42,5.47,5.63,5.34,5.46,5.3,
5.78,5.68,5.85) )
inits
list( theta = 5.48 )
```