1. Gamma Poisson mallissa, kun $p(y|\theta) = \text{Poisson}(\theta)$ ja $\pi(\theta) = \text{Gamma}(\alpha, \beta)$ joka on konjugaatti priori

$$\begin{aligned}&\pi(\theta|y_1, \dots, y_n) \sim \text{Gamma}(\alpha + y_1 + \dots + y_n, \beta + n)\\&= \text{Gamma}(300 + 409, 5 + 8) = \text{Gamma}(709, 13).\end{aligned}$$

Kolmannessa demo paperissa laskettiin osittaisintegroinnilla $\text{Gamma}(\alpha, \beta)$ jakauman kertymäfunktio silloin kun $\alpha = 2$. Samoin saadaan kun $\theta \sim \text{Gamma}(\alpha, \beta)$ ja $\alpha \in \mathbb{N}$

$P(\theta > t) = \sum_{k=0}^{\alpha-1} \frac{(t\theta)^k}{k!} \exp(-\theta t)$

Tätä voidaan ymmärtää myös seuraavasti: kun $\alpha \in \mathbb{N}$ $\theta \stackrel{\mathcal{L}}{=} \tau_1 + \dots + \tau_\alpha$ jossa τ_i ovat i.i.d β-eksponentiaalisia.

$P(\tau_1 + \dots + \tau_k > t) = P(N_t < k)$ jossa N_t on Poisson parametrilla βt.

$$\pi(\theta > 60|y_1, \dots, y_n) = \sum_{i=0}^{708} \frac{(780)^i}{i!} \exp(-780)$$

jossa $780 = k \times 60 = 13 \times 60$.

2. R- antaa tuloksen

```
> 1-pgamma(60,709,13)
[1] 0.004733205
```

3. Jos yksilöiden välissä ei ole riippuvuutta, oletetaan että jokaisella yksilöllä on oma parametri, ja Bayes inferenssi suoritetaan rippumattomasti:

$$\theta_i \sim \text{Gamma}(\alpha, \beta), i = 1, 2$$

ovat a priori (ja myös a posteriori) riippumattomia, ja

$$p(y_i|\theta_1, \theta_2) = \begin{cases} p(y_i|\theta_1) = \text{Poisson}(\theta_i) & \text{kun } 1 \le i \le 4 \\ p(y_i|\theta_2) = \text{Poisson}(\theta_2) & \text{kun } 5 \le i \le 8 \end{cases}$$

ja havainnot ovat ehdollisesti riippumattomia ehdolla parametetreja.

4. $\pi(\theta) \propto \mathbf{1}(\theta > \theta_0)$, $\theta_0 > 0$

ja $Y_1, \dots, Y_n$ ovat ehdollisesti riippumattomia ehdolla θ, samalla θ-eksponentiaalisella jakaumalla

$$\begin{aligned}&P(\theta|y_1, \dots, n) \propto \mathbf{1}(\theta > \theta_0)\theta^n \exp(-\theta s_n)\\&s_n = \sum_{i=1}^{n} y_i\end{aligned}$$

Tämä on Gamma$(n+1, s_n)$ jakauma ehdollistettu tapahtumaan $\{\theta > \theta_0\}$, Tehtävän datalla $\theta \sim$ Gamma$(17, 31.28)$ ehdollistettu väliin $[0.3, +\infty)$. Gamma$(17, 31.28)$ jakauman keskiarvo on $17/31.28 = 0.5434783$, sen moodi on pisteessä $(\alpha - 1)/\beta = 0.51150$.

Ehdollistetun jakauman keskiarvo voidaan laskea WINBUGSILLA,

mutta myös analyyttisesti: kun $\theta \sim$ Gamma(α, β),

$$E(\theta|\theta > t) = \frac{E(\theta\mathbf{1}(\theta > t))}{P(\theta > t)} = \frac{\int_t^\infty \theta\theta^{\alpha-1}\exp(-\beta\theta)d\theta}{\int_t^\infty \theta^{\alpha-1}\exp(-\beta\theta)d\theta}$$

jossa normalisointi vakiot supistuvat pois,

$$= \frac{\Gamma(\alpha+1)\beta^{-(\alpha+1)}P(\tau > t)}{\Gamma(\alpha)\beta^{-(\alpha)}P(\theta > t)}$$

jossa $\tau \sim \Gamma(\alpha + 1, \beta)$

$$= \frac{\alpha}{\beta}\frac{\sum_{k=0}^{\alpha}\frac{(t\theta)^k}{k!}\exp(-\theta t)}{\sum_{k=0}^{\alpha-1}\frac{(t\theta)^k}{k!}\exp(-\theta t)} = \frac{\alpha}{\beta}\left(1 + \frac{\frac{(t\theta)^\alpha}{\alpha!}}{\sum_{k=0}^{\alpha-1}\frac{(t\theta)^k}{k!}}\right)$$

5.

$$p(\text{ data }|\theta) =$$
$$\exp(-\theta 7 \times 4) \times (1 - \exp(-\theta))^{12} \times \theta^{\#\{y_i \in (1,4)\}}\exp(-\theta\sum_{y_i \in (1,4)} y_i)$$

Kun priori $\pi(\theta) \sim$ Gamma$(\theta; \alpha, \beta) \propto \theta^{\alpha-1}\exp(-\beta\theta)$, (kun $\alpha = \beta = 0$ $\pi(\theta) \propto \theta^{-1}$ on epäaito,

$$\pi(\theta|\text{ data}) \propto \pi(\theta)p(\text{ data }|\theta) \propto$$
$$\theta^{\alpha-1+17}\exp\left(-\theta\big(\beta + 7 \times 4 + \sum_{y_i \in (1,4)} y_i\big)\right)(1 - \exp(-\theta))^{12}$$
$$= \theta^{\widetilde{\alpha}-1}\exp(-\theta\widetilde{\beta})(1 - \exp(-\theta))^M$$
$$= \sum_{k=0}^{M}\binom{M}{k}(-1)^k\theta^{\widetilde{\alpha}-1}\exp(-\theta(\widetilde{\beta} + k))$$

jossa käytettiin Newtonin kaavaa. Periatteessä tätä pysty vielä integroimaan analyyttisesti:

$$\int_0^\infty \pi(\theta)p(\text{ data }|\theta)d\theta = \Gamma(\widetilde{\alpha})\sum_{k=0}^{M}\binom{M}{k}(-1)^k(\widetilde{\beta} + k)^{-1}$$

Vaihtoehtoisesti voidaan laskea winbugsilla:

model{ă

theta ∼ dgamma(alpha ,beta)

```
for( i in 1:17 ) {
Y[1]~ dexp(theta)
}
pleft<-(1-exp(-theta))
Yleft~dbin( pleft, 12 )
pright<-exp(-4*theta)
Yright~dbin( pright , 7 )
}
```

```
data
list( alpha=0.001, beta=0.001, Yleft=12,Yright=7,
Y=(c(0.30,4.51,0.15,7.54,0.90,0.30,1.56,2.72,0.29,4.08,0.49,0.28,2.67,1.46,4.03)
)
```

Tässä hujataan WINBUGS jotta saisimme sensuroidujen havaintojen uskottavuus mukaan:

$$\exp(-4 \times 7) = P(N = 7)$$

jossa N on binomialinen parametreilla $\exp(-4), 7$, ja vastaavasti

$$(1 - \exp(-1))^{12} = P(N = 12)$$

jossa N on binomialinen parametreilla $(1 - \exp(-1)), 12$.

6. Olkoon $\theta > 0$

$$p(y|\theta) = \theta y^{-(\theta+1)} \mathbf{1}(y > 1)$$

kirjoitetaan eksponentiaaliseen muotoon:

$$p(y|\theta) = \exp(-(\theta + 1)\log(y) - \log(\theta))\mathbf{1}(y > 1)$$

Kojugaatti jakauma θ:lle on

$$\pi_{\alpha,\beta}(\theta) \propto \exp(\alpha\theta - \beta\log(\theta)) =$$
$$\theta^{-\beta}\exp(\alpha\theta)$$

joka on Gamma$(-\beta + 1, -\alpha)$ jakauma silloin kun $-\beta + 1 > 0$ eli $\beta < 1$, ja $\alpha < 0$ Posteriori on muotoa

$$\pi_{\alpha,\beta}(\theta|y_1, \ldots, y_n) \sim \theta^{n-\beta}\exp\big(\theta(\alpha - \sum_{i=1}^{n}\log(y_i))\big)$$

$$\sim \text{Gamma}(n - \beta + 1, \sum_{i=1}^{n}\log(y_i) - \alpha)$$

7.

$$\pi(\theta)p(y_1,\ldots,y_n|\theta) \propto$$
$$\theta^{-(\alpha+1)}\exp\Big(-\frac{\beta}{\theta}\Big)\theta^{-n/2}\exp\Big(-\frac{\frac{1}{2}\sum_{i=1}^n(y_i-y_0)^2}{\theta}\Big)$$
$$\propto \theta^{-(\alpha+1+n/2)}\exp\Big(-\frac{\beta+\frac{1}{2}\sum_{i=1}^n(y_i-y_0)^2}{\theta}\Big)$$

joka on käänteisgamma jakauma

$$\pi(\theta|y_1,\ldots,y_n) = \frac{\widetilde{\beta}^{\widetilde{\alpha}}}{\Gamma(\alpha)}\theta^{-(\widetilde{\alpha}+1)}\exp\big(-\widetilde{\beta}/\theta\big)$$

parametreilla

$$\widetilde{\alpha} = \alpha + n/2\widetilde{\beta} = \beta + \frac{1}{2}\sum_{i=1}^n (y_i-y_0)^2$$

$$E(\theta|y_1,\ldots,y_n) = \int_0^\infty \theta\pi(\theta|y_1,\ldots,y_n)d\theta$$
$$= \frac{\widetilde{\beta}^{\widetilde{\alpha}}}{\Gamma(\alpha)}\int_0^\infty \theta^{-\widetilde{\alpha}}\exp\big(-\widetilde{\beta}/\theta\big)d\theta$$
$$= \frac{\widetilde{\beta}^{\widetilde{\alpha}}}{\Gamma(\alpha)}\frac{\Gamma(\widetilde{\alpha}-1)}{\widetilde{\beta}^{\widetilde{\alpha}-1}}$$
$$= \frac{\widetilde{\beta}}{\widetilde{\alpha}-1} = \frac{\beta+\frac{1}{2}\sum_{i=1}^n(y_i-y_0)^2}{\alpha+n/2-1}$$

Kun $\widetilde{\alpha} < 1$ $E(\theta|y_1,\ldots,y_n) = +\infty$.
Samoin

$$E(\theta^2|y_1,\ldots,y_n) = \int_0^\infty \theta^2\pi(\theta|y_1,\ldots,y_n)d\theta$$
$$= \frac{\widetilde{\beta}^{\widetilde{\alpha}}}{\Gamma(\alpha)}\int_0^\infty \theta^{-\widetilde{\alpha}+1}\exp\big(-\widetilde{\beta}/\theta\big)d\theta$$
$$= \frac{\widetilde{\beta}^{\widetilde{\alpha}}}{\Gamma(\alpha)}\frac{\Gamma(\widetilde{\alpha}-2)}{\widetilde{\beta}^{\widetilde{\alpha}-2}}$$
$$= \frac{\widetilde{\beta}^2}{(\widetilde{\alpha}-1)(\widetilde{\alpha}-2)} = \frac{\Big(\beta+\frac{1}{2}\sum_{i=1}^n(y_i-y_0)^2\Big)^2}{(\alpha+n/2-1)(\alpha+n/2-2)}$$

Kun $\widetilde{\alpha} < 2$ $E(\theta^2|y_1,\ldots,y_n) = +\infty$.

$$\text{Var }(\theta|y_1,\ldots,y_n) = \frac{\Big(\beta+\frac{1}{2}\sum_{i=1}^n(y_i-y_0)^2\Big)^2}{(\alpha+n/2-1)^2}\Big(\frac{(\alpha+n/2-1)}{(\alpha+n/2-2)}-1\Big)$$
$$= \frac{\Big(\beta+\frac{1}{2}\sum_{i=1}^n(y_i-y_0)^2\Big)^2}{(\alpha+n/2-1)^2}\frac{1}{(\alpha+n/2-2)}$$

käänteisgamma jakauman parametreilla $\widetilde{\alpha}, \widetilde{\beta}$ %90 Bayes HDI-luottamusväli, löytyy seuraavasti.

Olkoon $\phi \sim Gamma(\widetilde{\alpha}, \widetilde{\beta})$ jakautunut, $\theta = \phi^{-1}$

Etsitään pisteet $s < t$ jolla

$$P(\theta < t) - P(\theta < s) = P(\phi > 1/t) - P(\phi > 1/s) = 0.9$$

$$s^{-(\widetilde{\alpha}+1)} \exp(-\widetilde{\beta}/s) = t^{-(\widetilde{\alpha}+1)} \exp(-\widetilde{\beta}/t)$$

8. winbugs koodi:

```
model {ă
phi ~ dgamma(alpha,beta)
theta <-1/phi
for( i in 1:n ) {
Y[i] ~ dnorm( y0, phi )
}
}
```

Huomataan että winbugsin parametrisaatiolla kun $Y \sim$ dnorm(μ,ϕă) Y on gaussinen jolla $E(Y) = \mu$ ja $\mathrm{Var}(Y) = 1/\phi$.