Demo 1
Partial differential equations, 2021

1. Classify each of the following PDEs: linear transport equation, Laplace's equation, Poisson equation, nonlinear Poisson equation, heat equation, wave equation, p-Laplace equation, eikonal equation. Is the PDE linear, semilinear quasilinear or fully non-linear? What is the order of the PDE?

2. Let $u(x) = |x|^{\alpha}$, $x \in \mathbb{R}^N$, $x \neq 0$, where $\alpha \in \mathbb{R}$. Calculate $\Delta u(x)$, where $\Delta u = \sum_{i=1}^{N} \partial_{x_i} \partial_{x_i} u = \sum_{i=1}^{N} D_{ii} u$.

3. Suppose that $u \in C^1(\mathbb{R}^N)$ and $\varphi \in C_0^1(\mathbb{R}^N)$. Prove using Gauss-Green theorem that

$$\int_{\mathbb{R}^N} u \partial_k \varphi \, dx = - \int_{\mathbb{R}^N} \varphi \partial_k u \, dx, \quad k = 1, \ldots, N.$$

4. Consider the Neumann problem:

$$\begin{cases} \Delta u(x) = f(x) & x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x) = 0, & x \in \partial \Omega, \end{cases} \tag{1}$$

   where $\frac{\partial u}{\partial \nu} = Du \cdot \nu$ and ν is the exterior unit normal vector of Ω. Show that either the problem (1) does not have a solution or it has many solutions.

5. Is the equation $\Delta u = 0$ in divergence or non-divergence form? Let $u \in C^2$ and $Du \neq 0$, $p > 2$. How about

$$|Du|^{p-2} \left(\Delta u + (p-2) |Du|^{-2} \sum_{i,j=1}^{N} D_{ij} u D_i D_j u \right) = 0$$

   (cf. p-Laplace equation)

6. Let u be a continuous function in $\Omega \subset \mathbb{R}^N$. Suppose that

$$\int_{\Omega} u \varphi \, dx = 0$$

   for all $\varphi \in C_0^{\infty}(\Omega)$. Show that $u(x) = 0$ for all $x \in \Omega$.