1. Classify the PDEs.

   (a) Linear transport equation

   $$\partial_t u + b \cdot Du = 0$$

   linear, first order;

   (b) Laplace's equation

   $$\Delta u = 0$$

   linear, second order;

   (c) Poisson's equation

   $$\Delta u = f$$

   linear, second order;

   (d) p-Laplace equation

   $$\operatorname{div}(|Du|^{p-2} Du) = 0$$

   quasilinear, second order;

   (e) Heat equation

   $$\partial_t u - \Delta u = 0$$

   linear, second order;

   (f) Wave equation

   $$\partial_t \partial_t u - \Delta u = 0$$

   linear, second order;

   (g) Eikonal equation

   $$|Du|^2 = 1$$

   fully non-linear, first order;

   (h) Non-linear Poisson's equation

   $$-\Delta u = f(u)$$

   semi-linear, second order;

   (i) Navier-Stokes equation

   $$\begin{cases} \partial_t u_i - \Delta u_i + u \cdot Du_i = \partial_{x_i} p \\ \operatorname{div} u = 0 \end{cases}$$

   semi-linear, second order.

2. Let $u(x) = |x|^\alpha$, $x \in \mathbb{R}^N$, $x \neq 0$, where $\alpha \in \mathbb{R}$. Calculate $\Delta u(x)$.

Solution. Let

$$u(x) = |x|^\alpha = (x_1^2 + x_2^2 + \ldots + x_N^2)^{\frac{\alpha}{2}}.$$

If $x \neq 0$, then

$$D_i u(x) = \alpha \, |x|^{\alpha-2} \, x_i$$

and

$$D_{ii} u(x) = \alpha \, |x|^{\alpha-2} + \alpha(\alpha-2) \, |x|^{\alpha-4} \, x_i^2$$

for $i = 1, \ldots, N$. Therefore

$$\Delta u(x) = \sum_{i=1}^N D_{ii} u(x) = \alpha(N + \alpha - 2) \, |x|^{\alpha-2} .$$

□

3. Suppose that $u \in C^1(\mathbb{R}^N)$ and $\varphi \in C_0^1(\mathbb{R}^N)$. Prove that

$$\int_{\mathbb{R}^N} u D_k \varphi \, dx = -\int_{\mathbb{R}^N} \varphi D_k u \, dx, \quad k = 1, 2, \ldots, N.$$

Proof. Since $\varphi \in C_0^1(\mathbb{R}^N)$, there is $R > 0$ such that the support of φ is contained in $B(0, R)$, that is

$$\operatorname{spt}(\varphi) \subset B(0, R).$$

By the Gauss-Green theorem, we have

$$\int_{B(0,R)} D_k(u\varphi) \, dx = \int_{\partial B(0,R)} u \varphi \nu_k \, dS(x) = 0 \tag{1}$$

for $k = 1, 2, \ldots, N$, where ν is the outward normal unit vector and the second equality follows from the fact that

$$\varphi = 0 \quad \text{on } \partial B(0, R).$$

We rewrite (1) as

$$\int_{B(0,R)} u D_k \varphi \, dx = -\int_{B(0,R)} \varphi D_k u \, dx.$$

Since $\varphi \equiv 0$ outside of $B(0, R)$, this implies the claim. □

4. Consider the Neumann problem:

$$\begin{cases} \Delta u(x) = f(x), & x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x) = 0, & x \in \partial\Omega. \end{cases}$$

Show that if the problem has a solution, then it has many solutions.

Proof. If u is a solution, then so is $v := u + c$ for any $c \in \mathbb{R}$. $\square$

5. Is the equation $\Delta u = 0$ in divergence form? Let $u \in C^2$ and $Du \neq 0$, $p > 2$. How about

$$|Du|^{p-2} (\Delta u + (p-2) |Du|^{-2} \sum_{i,j=1}^{N} D_{ij} u D_i D_j u) = 0?$$

Solution. $\Delta u = \operatorname{div}(Du) = 0$. Second,

$$D_i(|Du|^{p-2} Du_i) = (p-2)/2(\sum_{j=1}^{N} |Du_j|^2)^{\frac{p-2}{2}-1} \sum_{j=1}^{N} 2D_j u D_{ij} u D_i u + |Du|^{p-2} D_{ii} u$$

for $i = 1, \ldots, N$, and so

$$\operatorname{div}(|Du|^{p-2} Du) = |Du|^{p-2} (\Delta u + (p-2) |Du|^{-2} \sum_{i,j=1}^{N} D_{ij} u D_i u D_j u.$$

$\square$

6. Let u be a continuous function in $\Omega \subset \mathbb{R}^N$. Suppose that

$$\int_\Omega u\varphi \, dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. Show that $u(x) = 0$ for all $x \in \Omega$.

Proof. Suppose on the contrary that $u(x_0) > 0$ for some $x_0 \in \Omega$. By continuity there exists $r > 0$ such that $u > 0$ in $B(x_0, 2r)$. Take a function $\varphi \in C_0^\infty(B(x_0, r))$ such that $\varphi(x_0) > 0$. Then

$$0 = \int_\Omega u\varphi \, dx = \int_{B(x_0,r)} u\varphi \, dx > 0,$$

which is a contradiction. The function φ can be defined by setting

$$\varphi(x) = \eta(\frac{x - x_0}{r}),$$

where

$$\eta(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 0. \end{cases}$$

$\square$