1. Solve

$$\begin{cases} (2,3)\cdot Du = 0, \\ u(0,y) = \sin y. \end{cases} \tag{1}$$

Proof. Fix a point $(x_0, y_0) \in \mathbb{R}^2$. We consider the curve

$$\Gamma_{x_0,y_0} = \{(\tilde{x}(s), \tilde{y}(s)) \in \mathbb{R}^2 : s \in \mathbb{R}\}$$

starting from (x_0, y_0), where

$$\begin{cases} \tilde{x}(s) = x_0 + 2s, \\ \tilde{y}(s) = y_0 + 3s. \end{cases}$$

Then by chain rule we have

$$\frac{d}{ds}u(\tilde{x}(s), \tilde{y}(s)) = (\tilde{x}'(s), \tilde{y}'(s)) \cdot Du(\tilde{x}(s), \tilde{y}(s)) = (2,3)\cdot Du(\tilde{x}(s), \tilde{y}(s)) = 0,$$

where the last identity follows from (1). Thus u is constant on the curve Γ_{x_0,y_0}. Given an arbitrary point $(x, y) \in \mathbb{R}^2$, we have $(x, y) \in \Gamma_{0,y-3x/2}$ since

$$\tilde{x}(x/2) = 0 + 2\cdot(x/2) = x \quad \text{and} \quad \tilde{y}(x/2) = (y - 3x/2) + 3\cdot(x/2) = y.$$

Thus by the initial condition in (1) we obtain

$$u(x,y) \overbrace{=}^{\substack{\text{u const. on} \\ \Gamma_{0,y-3x/2}}} u(0, y - 3x/2) = \sin(y - 3x/2).$$

□

2. Solve

$$\begin{cases} (2,3)\cdot Du = x, \\ u(0,y) = y. \end{cases} \tag{2}$$

Proof. Fix a point (x_0, y_0). Consider the same curve as in problem 1

$$\Gamma_{x_0,y_0} = \{(\tilde{x}(s), \tilde{y}(s)) \in \mathbb{R}^2 : s \in \mathbb{R}\}$$

starting from (x_0, y_0), where

$$\begin{cases} \tilde{x}(s) = x_0 + 2s, \\ \tilde{y}(s) = y_0 + 3s. \end{cases}$$

Denote $z(s) := u(\tilde{x}(s), \tilde{y}(s))$. Then by chain rule we have this time

$$\begin{aligned} \frac{d}{ds}z(s) = \frac{d}{ds}u(\tilde{x}(s), \tilde{y}(s)) &= (\tilde{x}'(s), \tilde{y}'(s)) \cdot Du(\tilde{x}(s), \tilde{y}(s)) \\ &= \tilde{x}(s) = x_0 + 2s. \end{aligned}$$

Thus by the fundamental lemma of calculus

$$z(s) = z(0) + \int_0^s x_0 + 2r\,dr = z(0) + sx_0 + s^2 \quad \text{for all } s \in \mathbb{R},$$

which by definition of z can be written as

$$u(\tilde{x}(s), \tilde{y}(s)) = u(x_0, y_0) + sx_0 + s^2 \quad \text{for all } s \in \mathbb{R}. \tag{3}$$

On the other hand, given an arbitrary point $(x, y) \in \mathbb{R}^2$, we have $(x, y) \in \Gamma_{0,y-3x/2}$ since

$$\tilde{x}(x/2) = 0 + 2 \cdot (x/2) = x \quad \text{and} \quad \tilde{y}(x/2) = (y - 3x/2) + 3 \cdot (x/2) = y.$$

Therefore from (3) we obtain

$$\begin{aligned} u(x, y) = u(\tilde{x}(x/2), \tilde{y}(x/2)) &= u(x_0, y_0) + sx_0 + s^2 \\ &= u(0, y - 3x/2) + 0 \cdot (x/2) + (x/2)^2 \\ &= y - 3x/2 + x^2/4, \end{aligned}$$

where the last identity follows from the initial condition in (2). □

3. Solve

$$\begin{cases} (2,3) \cdot Du = xu, \\ u(0, y) = y. \end{cases}$$

Proof. Fix a point $(0, y_0)$. Again we consider the same curve as in problem 1

$$\Gamma_{0,y_0} = \{(\tilde{x}(s), \tilde{y}(s)) \in \mathbb{R}^2 : s \in \mathbb{R}\}$$

starting from $(0, y_0)$, where

$$\begin{cases} \tilde{x}(s) = 2s, \\ \tilde{y}(s) = y_0 + 3s. \end{cases}$$

Denote $z(s) := u(\tilde{x}(s), \tilde{y}(s))$. Then by chain rule we have this time

$$\begin{aligned} \frac{d}{ds} z(s) = \frac{d}{ds} u(\tilde{x}(s), \tilde{y}(s)) &= (\tilde{x}'(s), \tilde{y}'(s)) \cdot Du(\tilde{x}(s), \tilde{y}(s)) \\ &= \tilde{x}(s) u(\tilde{x}(s), \tilde{y}(s)) = 2sz(s). \end{aligned}$$

This is solved by $z(s) = ce^{s^2}$ whenever $c \in \mathbb{R}$, so that

$$u(\tilde{x}(s), \tilde{y}(s)) = ce^{s^2} \quad \text{for all } s \in \mathbb{R}. \tag{4}$$

On the other hand, given an arbitrary point $(x, y) \in \mathbb{R}^2$, we have $(x, y) \in \Gamma_{0,y-3x/2}$ since

$$\tilde{x}(x/2) = 2 \cdot (x/2) = x \quad \text{and} \quad \tilde{y}(x/2) = (y - 3x/2) + 3 \cdot (x/2) = y.$$

Then by the initial condition and (4) we have

$$y - 3x/2 = u(0, y - 3x/2) = u(\tilde{x}(0), \tilde{y}(0)) = ce^{0^2} = c,$$

so that $c = y - 3x/2$. Thus

$$u(x, y) = u(\tilde{x}(x/2), \tilde{y}(x/2)) = ce^{(x/2)^2} = (y - 3x/2)e^{x^2/4}.$$

□

4. Solve

$$(1, x) \cdot Du = 0. \tag{5}$$

Proof. Fix a point $(x_0, y_0) \in \mathbb{R}^2$. We consider the curve

$$\Gamma_{x_0,y_0} = \left\{(\tilde{x}(s), \tilde{y}(s)) \in \mathbb{R}^2 : s \in \mathbb{R}^2\right\},$$

where

$$\begin{cases} \tilde{x}'(s) = 1, & \tilde{x}(0) = x_0, \\ \tilde{y}'(s) = \tilde{x}(s), & \tilde{y}(0) = y_0. \end{cases} \tag{6}$$

We set $z(s) = u(\tilde{x}(s), \tilde{y}(s))$. Then

$$\frac{d}{ds} z(s) = (\tilde{x}'(s), \tilde{y}'(s)) \cdot Du(\tilde{x}(s), \tilde{y}(s)) = (1, \tilde{x}(s)) \cdot Du(\tilde{x}(s), \tilde{y}(s)) = 0.$$

The solutions to the system (6) are

$$\begin{cases} \tilde{x}(s) = s + x_0, \\ \tilde{y}(s) = s^2/2 + x_0 s + y_0. \end{cases}$$

Given $(x, y) \in \mathbb{R}^2$, we have u constant on the curve $\Gamma_{x,y}$. Thus

$$\begin{aligned} u(x, y) = u(\tilde{x}(-x), \tilde{y}(-x)) &= u(-x + x, x^2/2 - x^2 + y) \\ &= u(0, y - x^2/2) \\ &=: f(y - x^2/2). \end{aligned}$$

In other words, any solution to (5) can be written as $u(x, y) = f(y - x^2/2)$ for some $f \in C^1(\mathbb{R})$. □

5. Let f and g be two given continuous functions and let c be a constant. Solve the initial value problem

$$\begin{cases} (f(y), 1) \cdot Du = cu, \\ u(x, 0) = g(x). \end{cases}$$

Proof. Let $(x_0, y_0) \in \mathbb{R}^2$. We consider the curve

$$\Gamma_{x_0,y_0} = \left\{(\tilde{x}(s), \tilde{y}(s)) \in \mathbb{R}^2 : s \in \mathbb{R}^2\right\},$$

where

$$\begin{cases} \tilde{x}'(s) = f(\tilde{y}(s)), & \tilde{x}(0) = x_0, \\ \tilde{y}'(s) = 1 & \tilde{y}(0) = x_0. \end{cases} \tag{7}$$

The solution to the system (7) are

$$\begin{cases} \tilde{x}(s) = \int_0^s f(r)\, dr + x_0 \\ \tilde{y}(s) = s + y_0. \end{cases}$$

We define $z(s) = u(\tilde{x}(s), \tilde{y}(s))$. Then

$$\begin{aligned}\frac{d}{ds} z(s) =& (\tilde{x}'(s), \tilde{y}'(s)) \cdot Du(\tilde{x}(s), \tilde{y}(s)) \\ =& (f(\tilde{y}(s)), 1) \cdot Du(\tilde{x}(s), \tilde{y}(s)) = cu(\tilde{x}(s), \tilde{y}(s)) = cz(s).\end{aligned}$$

This is solved by $z(s) = ae^{cs}$ whenever $a \in \mathbb{R}$, i.e.

$$u(\tilde{x}(s), \tilde{y}(s)) = ae^{cs} \quad \text{for all } s \in \mathbb{R}. \tag{8}$$

On the other hand an arbitrary $(x, y) \in \mathbb{R}^2$ lies on the curve Γ_{x_0, y_0}, where $x_0 = x - \int_0^y f(r)\, dr$ and $y_0 = 0$, since

$$\tilde{x}(y) = \int_0^y f(r)\, dr + x_0 = x, \quad \text{and} \quad \tilde{y}(y) = y + 0 = y.$$

Then in particular by the initial condition

$$u(\tilde{x}(0), \tilde{y}(0)) = u(x_0, y_0) = u(x - \int_0^y f(r)\, dr, 0) = g(x - \int_0^y f(r)\, dr)$$

and on the other hand by (8)

$$u(\tilde{x}(0), \tilde{y}(0)) = ae^{c \cdot 0} = a$$

so that $a = g(x - \int_0^y f(r)\, dr)$. Hence, using (8) again we obtain

$$u(x, y) = u(\tilde{x}(y), \tilde{y}(y)) = ae^{cy} = g(x - \int_0^y f(r)\, dr)e^{cy}.$$

□

6. Suppose that $u \in C^1(\mathbb{R}^2)$ is a solution to

$$(a(x, y), b(x, y)) \cdot Du = 0.$$

Show that for arbitrary $H \in C^1(\mathbb{R})$ also $H(u)$ is a solution.

Proof. By the chain rule

$$D(H(u)) = H'(u)Du.$$

Thus

$$\begin{aligned}(a(x, y), b(x, y)) \cdot D(H(u)) =& (a(x, y), b(x, y)) \cdot (H'(u)Du) \\ =& H'(u)((a(x, y), b(x, y)) \cdot Du) = 0\end{aligned}$$

i.e. $H(u)$ is also a solution.

□