Demo 3
Partial differential equations, 2021

1. Prove that if f is continuous, then

$$f(x) = \lim_{r\to 0} \fint_{B(x,r)} f\,dy = \lim_{r\to 0} \fint_{\partial B(x,r)} f\,dS,$$

where

$$\fint_{B(x,r)} f\,dy := \frac{1}{|B(x,r)|}\int_{B(x,r)} f\,dy,$$
$$\fint_{\partial B(x,r)} f\,dS := \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} f\,dS$$

is the average of f over $B(x,r)$ or $\partial B(x,r)$.

2. Find a solution to the following boundary value problem, $N \geq 3$

$$\begin{cases} \Delta u = 0 & \text{in } B(0,2)\setminus \overline{B}(0,1) \\ u = 0 & \text{on } \partial B(0,2), \\ u = 1 & \text{on } \partial B(0,1). \end{cases}$$

3. Find a solution to the following boundary value problem

$$\begin{cases} \Delta u = 1 & \text{in } B(0,1), \\ u = 0 & \text{on } \partial B(0,1). \end{cases}$$

4. Let $f \in C_0^\infty(\mathbb{R}^N)$,

$$u(x) := (\Phi * f)(x) := \int_{\mathbb{R}^N} \Phi(x-y) f(y)\,dy.$$

Show that $\partial_{x_i} u \in C(\mathbb{R}^N)$.

5. Prove that Laplace's equation

$$\Delta u = 0$$

is rotation invariant: if O is an orthogonal $N \times N$ matrix and we define

$$v(x) = u(Ox),$$

then

$$\Delta v = 0.$$

6. Let $u \in C^1(\mathbb{R}^N)$ and

$$\varphi(r) = \fint_{\partial B(x,r)} u(y)\,dS(y).$$

Prove that

$$\varphi'(r) = \fint_{\partial B(0,1)} Du(x+ry)\cdot y\,dS(y).$$