1. Prove that if f is continuous, then

$$f(x) = \lim_{r\to 0} \fint_{B(x,r)} f\,dy = \lim_{r\to 0} \fint_{\partial B(x,r)} f dS.$$

Proof. Since f is continuous, for any $\varepsilon > 0$ there exists $r > 0$ such that

$$|f(y) - f(x)| \le \varepsilon \quad \text{for all } y \in B(x,r).$$

Thus

$$\begin{aligned}\left|\fint_{B(x,r)} f(y)\,dy - f(x)\right| &= \left|\fint_{B(x,r)} f(y) - f(x)\,dy\right| \\ &\le \fint_{B(x,r)} |f(y) - f(x)|\,dy \\ &\le \fint_{B(x,r)} \varepsilon\,dy = \varepsilon.\end{aligned}$$

This means that $\lim_{r\to 0} \fint_{B(x,r)} f(x)\,dy = f(x)$. The proof of the second identity is completely analogical. □

2. When $N \ge 3$, find a solution to

$$\begin{cases} \Delta u = 0 & \text{in } B(0,2) \setminus \overline{B}(0,1) \\ u = 0 & \text{on } \partial B(0,2), \\ u = 1 & \text{on } \partial B(0,1). \end{cases} \tag{1}$$

Proof. Recall that when $N \ge 3$, the fundamental solution takes the form

$$\Phi(x) = c_N \frac{1}{|x|^{N-2}}$$

and that it solves

$$\Delta\Phi = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

The idea is to scale and lift Φ so that it solves the Dirichlet problem (1). That is, we set

$$u(x) := a\Phi(x) + c$$

for some $a, c \in \mathbb{R}$. Then $\Delta u = 0$ in $B(0,2) \setminus \overline{B}(0,1)$ and we need to take a and c so that

$$\begin{cases} u = 0 & \text{on } \partial B(0,2) \\ u = 1 & \text{on } \partial B(0,1) \end{cases}$$

i.e.

$$\begin{cases} \frac{ac_N}{2^{N-2}} + c = 0, \\ \frac{ac_N}{1^{N-2}} + c = 1. \end{cases}$$

This leads to $a = \frac{1-c}{c_N}$ so that

$$\begin{aligned} &\frac{(1-c)}{2^{N-2}} + c = 0 \\ \iff &c(1 - 2^{2-N}) = -2^{2-N} \\ \iff &c = \frac{2^{2-N}}{2^{2-N} - 1} \end{aligned}$$

and

$$a = c_N^{-1}(1 - \frac{2^{2-N}}{2^{2-N} - 1}) = -c_N^{-1}(\frac{1}{2^{2-N} - 1}).$$

□

3. Find a solution to

$$\begin{cases} \Delta u = 1 & \text{on } B(0,1), \\ u = 0 & \text{on } \partial B(0,1). \end{cases}$$

Proof. Suppose first that $N = 1$. Then $\Delta u = u'' = 1$ implies $u = ax^2 + c = \frac{1}{2}x^2 - \frac{1}{2}$. Suppose then that $N \geq 2$ and consider

$$u(x) = v(|x|) = v(r(x))$$

for some $v : [0, \infty) \to \mathbb{R}$, $r(x) = |x|$. Then (see derivation of the fundamental solution)

$$1 = \Delta u(x) = v''(r) + \frac{N-1}{r} v'(r),$$

i.e. v must solve

$$rv''(r) + (N-1)v'(r) = r \quad \text{for all } r \in [0, 1). \tag{2}$$

Inspired by the solution when $N = 1$, we guess that the solution has the form

$$v(r) = ar^2 + br + c$$

for some $a, b, c \in \mathbb{R}$. From (2) we obtain for all $r \in [0, 1)$

$$\begin{aligned} &2ar + (N-1)(2ar + b) = r \\ \iff &2arN + (N-1)b = r \end{aligned}$$

so that $a = \frac{1}{2N}$ and $b = 0$. To satisfy the boundary condition we take $c = -\frac{1}{2N}$. Thus the solution takes the form

$$u(x) = \frac{|x|^2}{2N} - \frac{1}{2N}.$$

□

4. Let $f \in C_0^\infty(\mathbb{R}^N)$

$$u(x) := (\Phi * f)(x) := \int_{\mathbb{R}^N} \Phi(x - y) f(y)\, dy.$$

Show that $\partial_{x_i} u \in C(\mathbb{R}^N)$.

Proof. See the first part of the proof of Theorem 4.6. □

5. Prove that Laplace's equation

$$\Delta u = 0$$

is rotation invariant: if Q is an orthogonal $N \times N$ matrix and we define

$$v(x) = u(Qx),$$

then $\Delta v(x) = 0$.

Proof. Let

$$v(x) = u(Qx).$$

We have (see below)

$$D^2 v(x) = Q^T D^2 u(Qx) Q. \tag{3}$$

Thus, since $QQ^T = I$, we get

$$\Delta v(x) = \mathrm{tr} D^2 v(x) = \mathrm{tr}(Q^T D^2 u(Qx) Q) = \mathrm{tr}(QQ^T D^2 u(Qx)) = \mathrm{tr} D^2 u(Qx) = \Delta u(Qx) = 0,$$

where we used the cyclic property of trace:

$$\mathrm{tr}(ABC) = \mathrm{tr}(CAB).$$

It remains to verify (3). Given a matrix A, denote by A_i the i:th row (so that $(A^T)_i$ is the i:th column). By the chain rule

$$Dv(x) = D(u(Qx)) = J_u(x) J_{Qx}(x) = Du(Qx)Q,$$

i.e. for $j = 1, \ldots, N$ we have

$$D_j v(x) = Du(Qx) \cdot Q_j^T = \sum_{k=1}^{N} D_k u(Qx)(Q_j^T)_k = \sum_{k=1}^{N} D_k u(Qx) Q_{kj}.$$

Thus

$$\begin{aligned}
D_{ij} v(x) = D_i(D_j v(x)) &= D_i(\sum_{k=1}^{N} D_k u(Qx) Q_{kj}) \\
&= \sum_{k=1}^{N} D_i(D_k u(Qx)) Q_{kj} && \Big|\text{chain rule again} \\
&= \sum_{k=1}^{N} (\sum_{l=1}^{N} D_l D_k u(Qx) Q_{li}) Q_{kj} \\
&= \sum_{k,l=1}^{N} D_{lk} u(Qx) Q_{li} Q_{kj} \\
&= (Q^T D^2 u(Qx) Q)_{ij},
\end{aligned}$$

where the last identity can be checked by computing out the matrix product. □

6. Let $u \in C^1(\mathbb{R}^N)$ and

$$\varphi(r) = \fint_{\partial B(x,r)} u(y) dS(y).$$

Prove that

$$\varphi'(r) = \fint_{\partial B(0,1)} Du(x+ry) \cdot y dS(y).$$

Proof. Rewrite φ by changing variables

$$\varphi(r) = \fint_{\partial B(0,1)} u(x+ry)\, dS(y).$$

Now fix $r > 0$ and let $h \in (0,1)$. Consider the difference quotient

$$\frac{\varphi(r+h) - \varphi(r)}{h} = \fint_{\partial B(0,1)} \frac{u(x+ry+hy) - u(x+ry)}{h}\, dS(y).$$

By mean value theorem there exists a vector $\xi \in [x+ry, x+ry+hy]$ such that

$$\frac{u(x+ry+hy) - u(x+ry)}{h} = Du(\xi) \cdot y.$$

Thus

$$\begin{aligned}
&\left| \fint_{\partial B(0,1)} \frac{u(x+ry+hy) - u(x+ry)}{h}\, dS(y) - \fint_{\partial B(0,1)} Du(x+ry) \cdot y\, dS(y) \right| \\
&= \left| \fint_{\partial B(0,1)} (Du(\xi) - Du(x+ry)) \cdot y\, dS(y) \right| \quad \Big| \xi \in \overline{B}_h(x+ry) \subset B_{r+1}(x) \\
&\leq \fint_{\partial B(0,1)} \sup_{\substack{\eta,\zeta \in B_{r+1}(x) \\ |\eta - \zeta| < h}} |Du(\eta) - D(\zeta)| \\
&\to 0
\end{aligned}$$

as $h \to 0$ since $u \in C^1(\mathbb{R}^N)$ implies that Du is uniformly continuous in $B_{r+1}(x)$. $\square$