Demo 4,
Partial Differential Equations, 2021

1. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies
$$-\Delta u \leq 0 \quad \text{in } \Omega.$$
Prove that
   (a)
$$u(x) \leq \fint_{B(x,r)} u(y)\, dy \quad \text{for all } B(x,r) \Subset \Omega,$$
   (b)
$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

2. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth and convex function. Assume that u is harmonic in Ω. Let $v = \varphi(u)$. Prove that
$$-\Delta v \leq 0 \quad \text{in } \Omega.$$

3. Assume that u is harmonic in Ω. Let $v = |Du|^2$. Prove that
$$-\Delta v \leq 0 \quad \text{in } \Omega.$$

4. Verify by a direct calculation that $x \mapsto \Phi(x)$ is harmonic in $\mathbb{R}^N \setminus \{0\}$.

5. Let $g \in \mathbb{R}$ and $f : [0, \infty) \to \mathbb{R}$ be continuous. Prove that if $u \in C^2(B_1) \cap C(\overline{B}_1)$ solves
$$\begin{cases} -\Delta u(x) = f(|x|) & \text{in } B_1, \\ u(x) = g & \text{on } \partial B_1, \end{cases}$$
then u is radially symmetric.

6. Prove that there is a unique solution $u \in C^2(B(0,1)) \cap C(\overline{B}(0,1))$ to the following boundary value problem
$$\begin{cases} \Delta u = u^3 & \text{in } B(0,1) \\ u = 0 & \text{on } \partial B(0,1) \end{cases}$$
applying the conclusion (b) of problem 1 in this exercise.