1. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$-\Delta u \leq 0 \quad \text{in } \Omega.$$

Prove that

(a)

$$u(x) \leq \fint_{B(x,r)} u(y)\, dy \quad \text{for all } B(x,r) \Subset \Omega,$$

(b)

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Proof. (a) Let $x \in \Omega$. Define

$$\phi(r) := \fint_{\partial B(x,r)} \phi(y)\, dS(y)$$

for all $0 < r < \operatorname{dist}(x, \partial\Omega)$. Then (see proof of Theorem 4.13)

$$\phi'(r) = \fint_{B(x,r)} \Delta u(y)\, dy \geq 0.$$

Thus by the fundamental theorem of calculus we have for any $\varepsilon \in (0, r)$

$$\fint_{B(x,r)} u(y)\, dy = \phi(r) = \phi(\varepsilon) + \int_{\varepsilon}^{r} \phi'(s)\, ds \geq \phi(\varepsilon) = \fint_{B(x,\varepsilon)} u(y)\, dy.$$

Since the right-hand side converges to $u(x)$ as $\varepsilon \to 0$, we obtain the claim.
(b) Suppose that there is $x_0 \in \Omega$ such that $u(x_0) = M$. Then

$$M = u(x_0) \leq \fint_{B(x,r)} u(y)\, dy.$$

This is possible only if $u \equiv M$ in $B(x,r)$. Since Ω is open and connected, it is path-connected. Thus we can connect any two points in Ω with a chain of balls, and so it follows that $u \equiv M$ in Ω. □

2. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth and convex function. Assume that u is harmonic in Ω. Let $v = \varphi(u)$. Prove that

$$-\Delta v \leq 0 \quad \text{in } \Omega.$$

Proof. Given vectors $\xi, \eta \in \mathbb{R}^N$, we denote by $\xi \otimes \eta$ the $N \times N$ matrix whose (i,j)-entry is $\xi_i \eta_j$. Then by chain rule

$$Dv = \varphi'(u) Du \quad \text{and} \quad D^2 v = \varphi''(u) Du \otimes Du + \varphi'(u) D^2 u.$$

Thus

$$\begin{aligned}
-\Delta v = -\mathrm{tr}(D^2 v) &= -\sum_{i=1}^{N} (\varphi''(u) D_i u D_i u + \varphi'(u) D_{ii} u) \\
&= -\sum_{i=1}^{N} \varphi''(u)(D_i u)^2 + \varphi'(u) D_{ii} u \\
&= -\varphi''(u) \left|Du\right|^2 - \varphi'(u)\Delta u \\
&\leq -\varphi' \Delta u. \\
&= 0.
\end{aligned}$$

□

3. Assume that u is harmonic in Ω. Let $v = \left|Du\right|^2$. Prove that

$$-\Delta v \leq 0 \quad \text{in } \Omega.$$

Proof. Recall that harmonic functions are smooth by Theorem 4.20. Therefore we can change the order of differentiation. Consequently for any $i \in \{1, \dots, N\}$ we have

$$\Delta(D_i u) = \sum_{k=1}^{N} D_{kk} D_i u = \sum_{k=1}^{N} D_i(D_{kk} u) = D_i \sum_{k=1}^{N} D_{kk} u = D_i(\Delta u) = 0,$$

so that $D_i u$ is harmonic. We now let $\varphi(s) = s^2$. Then φ is convex and smooth, and thus Problem 2 implies that

$$\Delta(\varphi(D_i u)) = 0,$$

i.e. $\varphi(D_i u)$ is harmonic. Consequently

$$-\Delta v = -\Delta(\sum_{i=1}^{N} (D_i u)^2) = \sum_{i=1}^{N} \Delta((D_i u)^2) = \sum_{i=1}^{N} \Delta(\varphi(D_i u)) = 0.$$

□

4. Verify by Direct calculation that

$$\Phi(x) = \begin{cases} c_2 \log(|x|), & N = 2, \\ c_N \left|x\right|^{2-N}, & N \geq 3, \end{cases}$$

is harmonic in $\mathbb{R}^N \setminus \{0\}$.

Proof. Denote $r(x) = |x|$ so that

$$Dr(x) = \frac{x}{|x|} \quad \text{and} \quad D^2 r(x) = \frac{I}{|x|} - \frac{x \otimes x}{\left|x\right|^3}.$$

Set also

$$\varphi(s) = \begin{cases} c_2 \log(s), & N = 2, \\ c_N s^{2-N}, & N \geq 3, \end{cases}$$

so that

$$\varphi'(s) = \begin{cases} \frac{c_2}{s}, & N = 2, \\ c_N(2-N)s^{1-N}, & N \geq 3, \end{cases}$$

and

$$\varphi''(s) = \begin{cases} -\frac{c_2}{s^2}, & N = 2, \\ c_N(2-N)(1-N)s^{-N}, & N \geq 3. \end{cases}$$

Then $\Phi(x) = \varphi(r(x))$ and

$$D^2\Phi(x) = \varphi''(r(x))Dr(x) \otimes Dr(x) + \varphi'(r(x))D^2r(x).$$

Thus

$$\begin{aligned} \Delta\Phi(x) &= \operatorname{tr}(\varphi''(|x|)\frac{x}{|x|} \otimes \frac{x}{|x|} + \varphi'(|x|)(\frac{I}{|x|} - \frac{x \otimes x}{|x|^3}) \\ &= \varphi''(|x|)\frac{|x|^2}{|x|^2} + \varphi'(|x|)(\frac{N}{|x|} - \frac{|x|^2}{|x|^3}) \\ &= \varphi''(|x|) + \varphi'(|x|)(\frac{N-1}{|x|}) \\ &= 0. \end{aligned}$$

□

5. Let $g \in \mathbb{R}$ and $f : [0, \infty) \to \mathbb{R}$ be continuous. Prove that if $u \in C^2(B_1) \cap C(\overline{B}_1)$ solves

$$\begin{cases} -\Delta u(x) = f(|x|) & \text{in } B_1, \\ u(x) = g & \text{on } \partial B_1, \end{cases} \tag{1}$$

then u is radially symmetric.

Proof. Fix $x_0 \in B_1 \setminus \{0\}$ and let Q be an orthogonal matrix such that $Qx_0 = |x_0| e_1$, where $e_1 = (1, 0, \ldots, 0)$. That is $x \mapsto Qx$ is a rotation of the coordinates that takes x_0 to $|x_0| e_1$. Define

$$v(x) := u(Qx).$$

In exercise 3, problem 5, we proved that for all $x \in B_1$ we have

$$D^2v(x) = Q^T D^2u(Qx)Q.$$

Thus, if $x \in B_1$, we have

$$-\Delta v(x) = -D^2u(Qx) = f(|Qx|) = f(|x|),$$

and if $x \in \partial B_1$, we have

$$v(x) = u(\underbrace{Qx}_{\in \partial B_1}) = g$$

That is, v is also a solution to problem (1). But then by uniqueness (Theorem 4.19) we must have $v \equiv u$ and in particular

$$u(x_0) = v(x_0) = u(Qx_0) = u(|x_0| e_1).$$

This implies the radial symmetry of u since x_0 was arbitrary. □

6. Prove that there is a unique solution $u \in C^2(B(0,1)) \cap C(\overline{B}(0,1))$ to the boundary value problem

$$\begin{cases} \Delta u = u^3 & \text{in } B(0,1), \\ u = 0 & \text{on } \partial B(0,1). \end{cases}$$

Proof. Clearly $v = 0$ is a solution. We need to show that it is unique. Suppose on the contrary that there exists a solution $u \in C^2(B(0,1)) \cap C(\overline{B}(0,1))$ that does not vanish identically in $B(0,1)$. We may assume that

$$E_+ := \{x \in B(0,1) : u(x) > 0\} \neq 0;$$

otherwise we consider $-u$. Let F be a connected component of E_+. Now we know that

$$-\Delta u = -u^3 < 0 \quad \text{in } F$$

and

$$u = 0 \quad \text{on } \partial F.$$

But then by conclusion (b) in problem 1, we have that

$$\max_{\overline{F}} u = \max_{\partial F} u = 0.$$

This is a contradiction since $u > 0$ in F. □