Demo 5,
Partial Differential Equations, 2021

1. Prove the following comparison principle: suppose that $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy $-\Delta v \leq -\Delta u$ in Ω. If $v \leq u$ on $\partial\Omega$, then $v \leq u$ in Ω.

2. Suppose that $u : \mathbb{R}^N \to \mathbb{R}$ is harmonic and bounded from above. Show that u is constant.
   Hint: set $v := M - u$, where $M = \sup_{\mathbb{R}^N} u$, and use Harnack's inequality.

3. Let u be a non-negative harmonic function in $\Omega = B(0,1) \setminus \{0\} \subset \mathbb{R}^N$. Show that there is a constant c, depending only on N, such that

$$\max_{\partial B(0,r)} u \leq c \min_{\partial B(0,r)} u,$$

   for all $0 < r \leq 1/2$.

4. Show that Green's function is non-negative.

5. Derive Green's function and Poisson kernel (i.e. $-\frac{\partial G(x,y)}{\partial \nu}$) for a unit ball when $N = 2$.
   Hint: $N \geq 3$ was considered in section 4.8 in the lectures.

6. Let u be a smooth solution, $N \geq 3$, of

$$\begin{cases} -\Delta u = f & \text{in } B(0,1) \subset \mathbb{R}^N, \\ u = g & \text{on } \partial B(0,1). \end{cases}$$

   Prove that

$$\max_{\overline{B}(0,1)} |u| \leq c \left(\max_{\partial B(0,1)} |g| + \max_{\overline{B}(0,1)} |f| \right),$$

   where $c > 0$ depends only on N.