1. Prove the following comparison principle: suppose that $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy $-\Delta v \leq -\Delta u$ in Ω. If $v \leq u$ on $\partial\Omega$, then $v \leq u$ in Ω.

Proof. Let $w = v - u$. Then we have

$$-\Delta w \leq 0 \quad \text{in } \Omega.$$

By problem 1 in the previous exercises, we have

$$\max_{\overline{\Omega}} w = \max_{\partial\Omega} w = \max_{\partial\Omega}(v-u) \leq 0.$$

Thus $v \leq u$ in Ω. □

2. Suppose that $u : \mathbb{R}^N \to \mathbb{R}$ is harmonic and bounded from above. Show that u is constant.

Proof. Let $M := \sup_{\mathbb{R}^N} u$. Then for any $\varepsilon > 0$ there is $x_\varepsilon \in \mathbb{R}^N$ such that

$$u(x_\varepsilon) \geq M - \varepsilon.$$

The function $x \mapsto M - u(x)$ is a non-negative harmonic function in $\mathbb{R}^N$. Thus by Harnack's inequality

$$\sup_{B(x_\varepsilon, r)} (M-u) \leq 3^N \inf_{B(x_\varepsilon, r)} (M-u) \leq 3^N \varepsilon,$$

i.e.

$$M - 3^N\varepsilon \leq u \leq M \quad \text{in } B(x_\varepsilon, r).$$

Since this holds for any $r > 0$ and ε was arbitrary, we see that $u \equiv M$ in $\mathbb{R}^N$. □

3. Let u be a non-negative harmonic function in $\Omega = B(0,1) \setminus \{0\} \subset \mathbb{R}^N$. Show that there is a constant c, depending only on N, such that

$$\max_{\partial B(0,r)} u \leq c \min_{\partial B(0,r)} u,$$

for all $0 < r \leq 1/2$.

Proof. There is a constant $k \in \mathbb{N}$, depending only on the dimension N, such that

$$\partial B(0,r) \subset \cup_{i=1}^k B(x_i, r/8),$$

where $x_i \in \partial B(0,r)$. Since $r < 1/2$, we can use Harnack's inequality in $B(x_i, r/4)$. Thus

$$\sup_{B(x_i, r/8)} u \leq 3^N \inf_{B(x_i, r/8)} u$$

and so

$$\max_{\partial B(0,r)} u \leq 3^{kN} \min_{\partial B(0,r)} u.$$

□

4. Show that Green's function is non-negative.

Proof. Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. Green's function in the region Ω is defined as $G : \Omega \times \Omega \to \mathbb{R}$,

$$G(x, y) := \Phi(y - x) - \varphi^x(y), \quad x, y \in \Omega, x \neq y,$$

where Φ is the fundamental solution to Laplace's equation and $\varphi^x : \Omega \to \mathbb{R}$ solves the Dirichlet problem

$$\begin{cases} \Delta\varphi^x(y) = 0, & y \in \Omega, \\ \varphi^x(y) = \Phi(y - x), & y \in \partial\Omega. \end{cases} \tag{1}$$

We fix $x \in \Omega$. Then it suffices to show that the function

$$u(y) := \Phi(y - x) - \varphi^x(y)$$

is non-negative in $\Omega\backslash\{x\}$. Since φ^x is bounded in Ω by the maximum principle and $\Phi(y-x) \to \infty$ as $y \to x$, we can take $r > 0$ so small that

$$u \geq 0 \quad \text{in } \overline{B}(x, r) \setminus \{x\}\,.$$

On the other hand, u is harmonic in $\Omega \setminus \overline{B}(x, r)$ and $u = 0$ on $\partial\Omega$. That is, we have

$$\Delta u = 0 \quad \text{in} \quad \Omega \setminus \overline{B}(x, r) \qquad \text{and} \qquad u \geq 0 \quad \text{on} \quad \partial(\Omega \setminus \overline{B}(x, r)).$$

Therefore it follows from the comparison principle that $u \geq 0$ also in $\Omega \setminus B(x, r)$. ☐

5. Derive Green's function and Poisson kernel (i.e. $-\frac{\partial G(x,y)}{\partial \nu}$) for a unit ball when $N = 2$.

Proof. The derivation is like in the lecture notes when $N = 2$. We have now

$$\varphi^x(y) = \Phi(|x|\,(y - x^*)) = c_2 \log((|x|\,(y - x^*)) = c_2 \log|x - y|\,,$$

where we used that $|x|\,(y - x^*) = |x - y|$ as shown in the lectures. So we still have

$$\begin{aligned} D_y G(x, y) &= D_y\Phi(y - x) - D_y\Phi(|x|\,(y - x^*)) \\ &= c_2\Big(\frac{y - x}{|y - x|^2} - \frac{|x|\,(|x|\,(y - x^*))}{||x|\,(y - x^*)|^2}\Big) \\ &= c_2\Big(\frac{y - x}{|y - x|^2} - \frac{|x|^2\, y - \overbrace{|x|^2\, x^*}^{=x}}{|y - x|^2}\Big) \\ &= \frac{c_2 y(1 - |x|^2)}{|y - x|^2}. \end{aligned}$$

The rest of the computation is then the same as in lectures. ☐

6. Let u be a smooth solution, $N \geq 3$, of

$$\begin{cases} -\Delta u = f & \text{in } B(0, 1) \subset \mathbb{R}^N, \\ u = g & \text{on } \partial B(0, 1). \end{cases}$$

Prove that

$$\max_{\overline{B}(0,1)} |u| \leq c \left(\max_{\partial B(0,1)} |g| + \max_{\overline{B}(0,1)} |f| \right),$$

where $c > 0$ depends only on N.

Proof. We have

$$u(x) = \int_{\partial B(0,1)} K(x,y)g(y)\,dS(y) + \int_{B(0,1)} G(x,y)f(y)\,dy,$$

where $K(x,y)$ is the Poisson kernel for $B(0,1)$

$$K(x,y) = \frac{1}{N\alpha_N}\frac{1-|x|^2}{|x-y|^N}$$

and $G(x,y)$ is the Green function for $B(0,1)$. Thus we have that

$$\begin{aligned}
|u(x)| &\leq \max_{\partial B(0,1)} |g| \overbrace{\int_{\partial B(0,1)} K(x,y)\,dS(y)}^{=1} + \max_{B(0,1)} |f| \int_{B(0,1)} G(x,y)\,dy \\
&\leq \max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f| \int_{B(0,1)} \Phi(x-y)\,dy \\
&\leq c(N)\big(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f|\big).
\end{aligned}$$

To see that $\int_{\partial B(0,1)} K(x,y)\,dS(y) = 1$, observe that $v = 1$ is a solution to

$$\begin{cases} \Delta v = 0 & \text{in } B(0,1), \\ v = 1 & \text{on } \partial B(0,1), \end{cases}$$

so that by Theorem 4.31 we have

$$1 = -\int_{\partial\Omega} 1\frac{\partial}{\partial\nu}G(x,y)\,dS(y) = \int_{\partial\Omega} K(x,y)\,dS(y).$$

□