# Demo 6,
# Partial Differential Equations, 2021

1. Denote by $B^+ := \{x = (x_1, x_2 \ldots, x_N) \in \mathbb{R}^N : |x| < 1, x_N > 0\}$ the open upper half ball. Assume that $u \in C^2(B^+) \cap C(\overline{B^+})$ is harmonic in $B^+$ with $u = 0$ on $\partial B^+ \cap \{x_N = 0\}$. Set

$$v(x) = \begin{cases} u(x) & \text{if } x_N \geq 0, \\ -u(x_1, \ldots, x_{N-1}, -x_N) & \text{if } x_N < 0, \end{cases}$$

for $x = (x_1, x_2, \ldots, x_N) \in B(0,1)$. Prove that $v$ is harmonic in $B(0,1)$.
Hint: Use $v$ as a boundary data to get a helpful harmonic function $w$.

2. Let $g \in C(\mathbb{R}^{N-1})$ be bounded. Show that

$$u(x) = \frac{2x_N}{N\alpha_N} \int_{\mathbb{R}^{N-1}} \frac{g(y)}{((x_1 - y_1)^2 + \cdots + (x_{N-1} - y_{N-1})^2 + x_N^2)^{N/2}} \, dy$$

is unique in the class of bounded, continuous solutions of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N_+, \\ u = g & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

3. Find the smallest eigenvalue for

$$\begin{cases} -u'' = \lambda u, & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

4. Calculate the smallest eigenvalue $\lambda_1(\Omega)$ of $-\Delta$ for

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\right\}.$$

Hint: Consider $u(x, y) = \sin(\pi x)\sin(\pi y)$.

5. Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$. Suppose that the following Neumann problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

has a non-trivial solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Show that $\lambda \geq 0$.

6. Suppose that $u$ is a smooth solution of the heat equation

$$\partial_t u - \Delta u = 0$$

in $\mathbb{R}^N \times (0, \infty)$. Show that $v(x,t) = u(\lambda x, \lambda^2 t)$ is also a solution for any $\lambda \in \mathbb{R}$.