Demo 7,
Partial Differential Equations, 2021

1. Write down an explicit formula for a solution of

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{R}^N \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^N \times \{t = 0\}, \end{cases}$$

with suitable assumptions on f, g.

2. Write down an explicit formula for a solution of

$$\begin{cases} \partial_t u - \Delta u + cu = f & \text{in } \mathbb{R}^N \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^N \times \{t = 0\}, \end{cases}$$

where $c \in \mathbb{R}$, and f and g satisfy suitable assumptions.

3. Let $E(0, 0, r)$, $r > 0$ be the heat ball. Find

$$\inf \{s < 0 : (0, s) \in E(0, 0, r)\}.$$

4. Let u be a solution to the heat equation in $\mathbb{R}^N \times (0, T)$. Show that

$$v(x, t) = u(x, t) - \frac{\mu}{(T + \varepsilon - t)^{N/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}},$$

$\mu \in \mathbb{R}$, $\varepsilon > 0$, is also a solution to the heat equation in $\mathbb{R}^N \times (0, T)$.

5. Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume that u solves the heat equation (i.e. $\partial_t u - \Delta u = 0$) and $v = \phi(u)$. Prove that v is a subsolution to the heat equation (i.e. $\partial_t v - \Delta v \leq 0$).

6. Comparison principle: Assume that u and v are solutions to the heat equation and $u \leq v$ on $\partial_p \Omega_T$, Ω a bounded domain. Show that

$$u \leq v \quad \text{in } \Omega_T.$$