Demo 8
Partial differential equations, 2021

1. Prove strong maximum principle in $B(0,1)\times(0,T)$ for the heat equation using Harnack's inequality.

2. Let u be a smooth solution of the following initial-boundary problem

$$\begin{cases} \partial_t u - \Delta u = u & \text{in } \Omega_T, \\ u = g & \text{on } \partial_p \Omega_T, \end{cases}$$

where Ω is a bounded smooth domain, $T>0$, and g a continuous function. Prove that

$$|u(x,t)| \le e^t \max_{\partial_p \Omega_T} |g| \quad \text{for all } (x,t)\in\Omega_T.$$

3. Show by a formal computation (without writing down the explicit solution) that if $u(x,t,s)$ solves

$$\begin{cases} \partial_t u(x,t,s) - \Delta u(x,t,s) = 0, & (x,t)\in\mathbb{R}^N\times(s,\infty), \\ u(x,s,s) = f(x,s), & (x,t)\in\mathbb{R}^N\times\{t=s\}, \end{cases}$$

(meaning that u as a function of (x,t) solves the equation with the starting time s), then

$$u(x,t) = \int_0^t u(x,t,s)\,ds$$

solves

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) = f(x,t), & (x,t)\in\mathbb{R}^N\times(0,\infty), \\ u(x,0) = 0 & (x,t)\in\mathbb{R}^N\times\{t=0\}. \end{cases}$$

4. Find a solution to the diffusion equation on the half-line with Dirichlet boundary condition

$$\begin{cases} \partial_t u - \partial_{xx} u = 0, & x>0, t>0, \\ u(x,0) = \varphi(x), & x>0, \\ u(0,t) = 0, & t>0, \end{cases}$$

where φ is a smooth bounded function with $\varphi(0)=0$.

5. Let $g\in C(\mathbb{R}^N)$ be a compactly supported smooth function. Show that the solution to

$$\begin{cases} \partial_t u = \Delta u & \text{in } \mathbb{R}^N\times(0,T), \\ u = g & \text{on } \mathbb{R}^N, \end{cases}$$

with $|u(x,t)|\le Ae^{a|x|^2}$ for some $a, A>0$, has the following conservation property:

$$\int_{\mathbb{R}^N} u(x,t)\,dx \equiv C \quad \text{for all } t>0.$$

6. Show that the direction of time plays no role in the wave equation (unlike the heat equation) i.e. that if $u(x,t)$ is a solution, also $u(x,-t)$ is.