PARTIAL DIFFERENTIAL EQUATIONS 2021, LECTURE NOTES

These lecture notes are essentially the PDE 2019 notes by Mikko Parviainen (with some minor changes by the current lecturer, Jarkko Siltakoski). See also "Partial Differential Equations" by L. C. Evans. These notes will be updated as the course progresses.

CONTENTS

1. Introduction

A partial differential equation (PDE), is an equation of an unknown function of two or more variables and its partial derivatives.

Example 1.1 (A very simple PDE)**.** Let $\Omega \subset \mathbb{R}^2$ be an open set. A function $u \in C^2(\Omega)$ is said to be a solution to the equation

$$D_{11}u(x_1,x_2) + D_{22}u(x_1,x_2) = 0 \quad \text{in } \Omega \tag{1.1}$$

if the equation holds for all $(x_1,x_2) \in \Omega$. One may now ask that what kind of functions are solutions and what kind of special properties they have.

Of course, there is an infinite number of solutions to equation (1.1) since any affine function is a solution (why?). To have uniqueness of solutions, one often needs the function to satisfy something more than just the PDE. A typical case is the boundary value problem, such as

$$\begin{cases} D_{11}u(x_1,x_2) + D_{22}u(x_1,x_2) = 0 & \text{in } B_1, \\ u(x_1,x_2) = g(x_1,x_2) & \text{on } \partial B_1, \end{cases}$$

where g is some fixed function on ∂B_1.

Depending on what we are modeling, the unknown u may describe a physical quantity such as heat or electric potential. Partial differential equations have a great variety of applications to mechanics, engineering electrostatics, quantum mechanics and many other fields of physics as well as finance. In addition, PDEs have a rich mathematical theory and their study can be also motivated from a purely mathematical perspective.

Example 1.2. We consider the initial value problem

$$\begin{cases} \partial_t u + b \cdot Du = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^N \times \{t = 0\}, \end{cases}$$

where

$$\begin{aligned}
&u:\mathbb{R}^N\times(0,\infty)\to\mathbb{R} \text{ (unknown)},\\
&g:\mathbb{R}^N\to\mathbb{R} \text{ (given)},\\
&b=(b_1,\dots,b_N) \text{ (given)},\\
&Du=(D_1u,\dots,D_Nu) \text{ (spatial gradient)}.
\end{aligned}$$

This is called a transport equation. Roughly, the reason for the name of the equation is as follows. Consider a conveyor belt that is for simplicity modelled in 1D, and infinity long. Then denote the mass density kg/m at x at time t by $u(x,t)$. The speed of the belt is b and thus mass exiting at $x+h$ in a short time s is approximately

$$-sbu(x+h,t)$$

and similarly mass entering at x is $sbu(x,t)$. Then it holds that

$$\begin{aligned}
&\text{change of mass on } [x,x+h] \text{ over time } [t,t+s]\\
&=(\text{mass entering at } x \text{ over time } [t,t+s])-(\text{mass exiting at } x+h \text{ over time } [t,t+s]).
\end{aligned}$$

That is

$$\int_x^{x+h}u(y,t+s)\,dy-\int_x^{x+h}u(y,t)\,dy\approx u(x,t)bs-u(x+h,t)bs$$

and so

$$\frac{1}{h}\int_x^{x+h}\frac{u(y,t+s)-u(y,t)}{s}\,dy\approx\frac{u(x,t)-u(x+h,t)}{h}b.$$

Since $\frac{1}{h}\int_x^{x+h}$ is just the integral average, we let $s,h\to 0$ to obtain

$$\partial_t u(x,t)=-\partial_x u(x,t)b.$$

What we naturally need to solve for mass density at given location x and time t is the initial mass density $g(x)$. We can guess that the solution is

$$u(x,t)=g(x-bt).$$

The conveyor belt example makes sense even for $g\notin C^1$, so already such an example suggests a need for "weak solutions". They are dealt in the later courses (PDE2, Viscosity theory).

1.1. **Notation.** Basic notation
$\mathbb{R}^N$, N-dimensional Euclidean space
$\mathbb{R}^1=\mathbb{R}$
$e_1=(1,0,\dots,0)$, ...,$e_N=(0,\dots,N)$, standard basis vectors
Ω, $U\subset\mathbb{R}^N$ open set, bounded unless otherwise stated
$|x|=\sqrt{x_1^2+\dots x_N^2}$ for $x\in\mathbb{R}^N$,
$\partial\Omega$ boundary of a set Ω,
$B(x,r)$ a ball of radius r centered at x
$|B(x,r)|=\alpha_N r^N=$ volume of a ball
$|\partial B(x,r)|=\omega_N r^{N-1}=$ area of a sphere
$\fint_{B(0,\varepsilon)}\dots\,dy=\frac{1}{|B(0,\varepsilon)|}\int_{B_{(0,\varepsilon)}}\dots\,dy$ mean value integral over ball
$\fint_{\partial B(0,\varepsilon)}\dots\,dy=\frac{1}{|\partial B(0,\varepsilon)|}\int_{B_{(0,\varepsilon)}}\dots\,dy$ mean value integral over sphere
$\Omega\Subset U$, $\overline{\Omega}\subset U$ and $\overline{\Omega}$ is compact

Functions and derivatives. $f : \Omega \to \mathbb{R}$, a function
$\operatorname{spt} f = \overline{\{x \in \Omega : f(x) \neq 0\}}$ = the support of f
$\frac{\partial u}{\partial x_j}(x) = \lim_{h\to 0} \frac{u(x+he_j)-u(x)}{h}$ partial derivative of u to the direction e_j
$u_{x_j} = D_j u = \frac{\partial u}{\partial x_j}$ shorthands for partial derivatives
$u_{x_i x_j} = D_{ij}u = \frac{\partial^2 u}{\partial x_i \partial x_j}$ higher order derivatives
$Du = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}) = (D_1 u, \dots, D_N u)$ gradient
$\frac{\partial u}{\partial \nu} = Du \cdot \nu$, outward normal derivative, ν outward unit normal vector

Multi-indexes and spaces. $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ multi-index
$|\alpha| = \alpha_1 + \ldots + \alpha_N$
$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_1^{\alpha_N}} u$
$D^k u(x) = \{D^\alpha u(x) : |\alpha| = k\}$, whenever $k \in \mathbb{N}$
$D^2 u(x) = \begin{pmatrix} D_{11}u(x) & \cdots & D_{1N}u(x) \\ \vdots & \ddots & \vdots \\ D_{N1}u(x) & \cdots & D_{NN}u(x) \end{pmatrix}$ Hessian matrix
$D^1 u(x) = Du(x)$
$C(\Omega) = \{f : f \text{ continuous in } \Omega\}$
$C(\overline{\Omega}) = \{f : f \text{ uniformly continuous on bounded subsets of } \Omega\}$
$C_0(\Omega) = \{f \in C(\Omega) : \operatorname{spt} f \Subset \Omega\}$
$C^k(\Omega) = \{f \in C(\Omega) : f \text{ is } k \text{ times continuously differentiable}\}$
$C^k(\overline{\Omega}) = \left\{u \in C^k(\Omega) : D^\alpha u \text{ is uniformly cont on all bounded subsets of } \Omega \text{ for } |\alpha| \leq k\right\}$
$C_0^k(\Omega) = C^k(\Omega) \cap C_0(\Omega)$
$C_0^\infty(\Omega) = \cap_{k=1}^\infty C^k(\Omega)$ = smooth functions
$C_0^\infty(\Omega) = C^\infty(\Omega) \cap C_0(\Omega)$ = compactly supported smooth functions
$\|f\|_{L^\infty(\Omega)} = \sup_\Omega |f|$ for $f \in C(\Omega)$.

1.2. **General form of a PDE and classifications.**

Definition 1.3 (General form)**.** Given a real valued function F, the expression of the form

$$F(D^k u(x), D^{k-1}u(x), \dots, Du(x), x) = 0$$

is the kth-order PDE, i.e. k is the highest order derivative. The unknown is a function $u : \Omega \to \mathbb{R}$.

Example 1.4. Most of the examples on this course are of second order. Let

$$\begin{aligned} &D^2 u(x) = \begin{pmatrix} D_{11}u & D_{12}u \\ D_{21}u & D_{22}u \end{pmatrix}, \\ &F : \mathbb{R}^{2\times 2} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}, \\ &F(X, \eta, u, x) := X_{11} + X_{22}, \\ &X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}. \end{aligned}$$

Then we have the Laplace's equation

$$F(D^2 u(x), Du(x), u(x), x) = D_{11}u(x) + D_{22}u(x) = 0.$$

Remark 1.5. Recall that

$$u \in C^k(\Omega) \iff D^\alpha u \in C(\Omega)$$

for any multi-index $\alpha \in (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$ such that $|\alpha_k| = \alpha_1 + \ldots + \alpha_k \le k$, where

$$D^\alpha u := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}} u$$

and

$$\frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}} u = \underbrace{\frac{\partial}{\partial x_i} \cdots \frac{\partial}{\partial x_i}}_{=\alpha_i \text{ times}} u.$$

Definition 1.6. If a PDE can be written in the forms below, then it is called

(1) linear, if
$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u(x) = f(x),$$
(2) semilinear, if
$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u(x) + a_0(D^{k-1}u, \dots, Du, u, x) = 0,$$
(3) quasilinear, if
$$\sum_{|\alpha| = k} a_\alpha(D^{k-1}u, \dots Du, u, x) + a_0(D^{k-1}u, \dots, Du, u, x) = 0,$$
(4) Fully nonlinear, if the PDE depends nonlinearly on the highest-order derivatives.

Remark 1.7. In the second order case we get

(1) linear if
$$Lu(x) := \sum_{i,j=1}^N a_{ij}(x) D_{ij} u(x) + \sum_{i=1}^N b_i(x) D_i u(x) + c(x) u(x) = f(x)$$
for given coefficients a_{ij}, b_i and c.
(2) Quasilinear, if
$$\sum_{i,j=1}^N a_{ij}(Du, u, x) D_{ij} u(x) + a_0(Du, u, x) = 0$$

Remark 1.8.

(1) In the linear case the LHS of PDE can be seen as a linear operator in the function space
$$L(au + bv) = aL(u) + bL(v),$$
where $a, b \in \mathbb{R}$ and u,v are functions (=linearity, L like linear), and PDE reads as
$$Lu = f.$$
Observe that the operator is linear, but naturally if there is a right hand side i.e. $\Delta u = f$, $\Delta v = f$, then
$$\Delta(u + v) = \Delta u + \Delta v = 2f,$$
so $u + v$ does not solve the same equation.
(2) Quasilinear equation is linear in the highest order derivatives.

Example 1.9.

(1) Laplacian i.e. Δu is linear. Let $a_{ij} = 0$ if $i \neq j$ and $a_{ij} = 1$ if $i = j$. Then

$$\Delta u = \sum_{i=1}^{N} D_{ii}u = \sum_{i,j=1}^{N} a_{ij}D_{ij}u$$

and

$$\Delta(au+bv) = \sum_{i=1}^{N} D_{ii}(au+bv) = a\sum_{i=1}^{N} D_{ii}u + b\sum_{i=1}^{N} D_{ii}u = a\Delta u + b\Delta v.$$

(2) $\Delta u + |Du|^2$ is semilinear.

Remark 1.10. There are further classifications. If the highest order term can be written in the form

$$\operatorname{div}(\mathscr{A}(D^{k-1}u,\dots,u,x)),$$

then the equation is in divergence form. If not, then it is in non-divergence form.

Example 1.11. Observe that

$$-\operatorname{div}(\mathscr{A}(x)Du) = -\sum_{i=1}^{N} D_i\Big(\sum_{j=1}^{N} a_{ij}(x)D_ju(x)\Big) = -\sum_{i,j=1}^{N} D_i(a_{ij}(x)D_ju(x)),$$

where $\mathscr{A}$ is a matrix with the entries a_{ij}. That is, a linear equation in the divergence form reads as

$$-\sum_{i,j=1}^{N} D_i(a_{ij}(x)D_ju(x)) + \sum_{i=1}^{N} b_i(x)D_iu(x) + c(x)u(x) = f(x).$$

Remark 1.12. There are further classifications. In particular:

- Elliptic="Laplace equation like"
- Parabolic="Heat equation like, time dependent"
- Hyperbolic="Wave equation like, time dependent"

One could give more precise statements, but we do not pursue this direction.

Remark 1.13. There are several boundary value problems. The most common in this course is the Dirichlet boundary value problem: the value of the function is given at the boundary

$$u = g \quad \text{on } \partial\Omega.$$

C.f. the derivation of the minimal surface equation.

We also encounter the Neumann problem, where the outward normal derivative is given:

$$\frac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega,$$

where $\frac{\partial u}{\partial \nu} = Du\cdot\nu$ is the outward normal derivative and ν is the outward unit normal vector.

Remark 1.14. There are several kind of solutions. On this course we consider classical solutions. It means that solution is smooth enough so that the derivatives in the equation make sense. For example, $u \in C^2(\Omega)$ such that $\Delta u = 0$ is a classical solution to the Laplace equation.

- Weak (distributional) solutions are considered in the course PDE2. Divergence form equations.
- Viscosity solutions are considered in the course Viscosity theory (PDE3). Control and game theory applications, probability and finance.
- Strong solutions, variational solutions...

Remark 1.15 (A well-posed problem). A PDE is well-posed if it has

(1) existence
(2) uniqueness
(3) stability: the solution depends continuously on data. In many cases in physics, data comes from measurements and it is crucial that small variations in the measurements only cause small change in the solution.

1.3. **Examples.**

Example 1.16.

(1) Laplace equation
$$\Delta u = \sum_{i=1}^{N} D_{ii} u = 0$$
(2) Poisson equation
$$-\Delta u(x) = f(x)$$
(3) Nonlinear Poisson equation (f not linear)
$$-\Delta u(x) = f(u)$$
(4) Heat equation
$$\partial_t u - \Delta u = 0$$
(5) Wave equation
$$\partial_{tt}^2 - \Delta u = 0$$
(6) Linear transport equation
$$\partial_t u + b \cdot Du = 0$$
(7) Eikonal equation
$$|Du|^2 = 1$$
(8) Eigenvalue equation or Helmholz equation
$$-\Delta u = \lambda u$$
(9) p-Laplace equation
$$-\operatorname{div}(|Du|^{p-2} Du) = 0, p > 1$$
(10) Infinity Laplace equation
$$\Delta_\infty u := \sum_{i,j=1}^{N} D_{ij}^2 u D_i u D_j u = 0$$
(11) Monge-Ampére equation
$$\det(D^2 u) = f$$
(12) Hamilton-Jacobi equation
$$\partial_t u + H(Du, x) = 0$$

(13) Parabolic p-Laplace/p-parabolic equation

$$\partial_t u = \operatorname{div}(|Du|^{p-2} Du)$$

(14) Porous medium equation

$$\partial_t u = \Delta u^m$$

(15) Minimal surface equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$$

(16) Navier-Stokes equation (system, $N = 3$) (1 million \$ prize)

$$\begin{cases} \partial_t(u_i) + u \cdot Du_i - \nu \Delta u_i = & -\frac{\partial p}{\partial x_i}, i = 1,2,3, \\ \operatorname{div} u = 0, u = (u_1, u_2, u_3) \end{cases}$$

Systems (many equations) are often, like in this case, more involved.

Next we will derive an equation for the soap film (minimal surface equation). Recall the following counterparts of the integration by parts from earlier courses. As a reminder, $\partial\Omega \in C^1$ roughly means that the boundary can be locally presented as a C^1 function. This suffices to ensure that the normal vector is well defined.

Theorem 1.17 (Gauss-Green theorem)**.** *Let $\partial\Omega \in C^1$ and $u \in C^1(\overline{\Omega})$. It holds that*

$$\int_\Omega D_i u \, dx = \int_{\partial\Omega} u \nu_i \, dS, \quad i = 1, 2, \ldots, N,$$

where $\nu = (\nu_1, \ldots, \nu_N)$ is the unit normal vector.

Example 1.18. In 1D the previous theorem is just the fundamental theorem of calculus

$$\int_a^b u' \, dx = u(b) - u(a).$$

From the Gauss-Green theorem, we obtain (ex).

Theorem 1.19 (Divergence theorem)**.** *Let $\varphi \in C_0^\infty(\Omega)$ and $F : \mathbb{R}^N \to \mathbb{R}^N$, $F_i \in C^1(\Omega)$. Then*

$$\int_\Omega F \cdot D\varphi \, dx = -\int_\Omega \operatorname{div} F \varphi \, dx, \quad i = 1, 2, \ldots, N.$$

Example 1.20. In 1D this is just integration by arts with zero boundary values

$$\int_a^b F\varphi' \, dx = 0 - 0 - \int_a^b F' \varphi \, dx.$$

Example 1.21 (Minimal surface equation)**.** Suppose you dip a wire frame into a soap solution, forming a soap film. The soap film tends to minimize the area i.e. it forms a minimal surface with boundary values fixed at the wire. Let $\Omega \subset \mathbb{R}^2$,

$$u : \Omega \to \mathbb{R}, \text{ unknown, the height of the soap film.}$$

Area of $3D$-surface $z = u(x)$ is

$$A(u) = \int_\Omega \sqrt{1+|Du|^2} \, dx,$$

where

$$|Du|^2 = \sum_{i=1}^{2} |D_i u|^2 .$$

Heuristically the idea is that since u is the minimizer, if we vary it a bit while preserving the boundary values, and compute a suitable derivative, then this derivative should be zero by the minimizing property. Let $\varphi \in C_0^\infty(\Omega)$. Since

$$\begin{aligned}\frac{d}{d\varepsilon}|D(u+\varepsilon\varphi)|^2 =& \frac{d}{d\varepsilon}\sum_{i=1}^{N}(D_i u+\varepsilon D_i\varphi)^2 \\ =& \sum_{i=1}^{N} 2(D_i u+\varepsilon D_i\varphi)D_i\varphi = 2(Du+\varepsilon D\varphi)\cdot D\varphi,\end{aligned}$$

we get

$$\begin{aligned}\frac{d}{d\varepsilon}A(u+\varepsilon\varphi) &= \frac{d}{d\varepsilon}\int_\Omega \sqrt{1+|D(u+\varepsilon\varphi)|^2}\,dx \\ &= \frac{d}{d\varepsilon}\int_\Omega \frac{2(Du+\varepsilon D\varphi)\cdot D\varphi}{2\sqrt{1+|D(u+\varepsilon\varphi)|^2}}\,dx.\end{aligned}$$

As the soap film minimizes the area, the solution u should satisfy for any perturbation $\varphi \in C_0^\infty(\Omega)$

$$\begin{aligned}0 = \frac{d}{d\varepsilon}A(u+\varepsilon\varphi)\Big|_{\varepsilon=0} &= \int_\Omega \frac{Du}{\sqrt{1+|Du|^2}}\cdot D\varphi\,dx \\ &\overset{\text{div-thm}}{=} -\int \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)\varphi\,dx.\end{aligned}$$

Since this holds for all $\varphi \in C_0^\infty(\Omega)$, we have (ex)

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0.$$

2. First order linear equations

We will solve some simple equations.

2.1. An equation with constant coefficients.

(1) ODE:

$$u:\mathbb{R}\to\mathbb{R},$$

$$\begin{cases}\frac{du}{dx} = 0,\\ u(0) = 1.\end{cases}$$

Then $u(x) = c$ and since $u(0) = 1$ the solution is $u(x) = 1$.

(2) PDE:

$$u:\mathbb{R}^2\to\mathbb{R},$$

$$\frac{\partial u(x,y)}{\partial x} = 0.$$

The solution is constant along horizonal lines i.e. $\{(x,y) : y = c\}$ is a characteristic curve of u for any $c \in \mathbb{R}$. Therefore $u(x,y) = f(y)$. Thus if we are given for example the initial condition $u(0,y) = y^2$, we get the whole solution

$$u(x,y) = y^2.$$

(3) Consider

$$\begin{aligned}
&u : \mathbb{R}^2 \to \mathbb{R}, \\
&a,b \in \mathbb{R},\ a \neq 0 \text{ or } b \neq 0, \\
&u : \mathbb{R}^2 \to \mathbb{R}.
\end{aligned}$$

(a) Geometric method. Suppose that we have the equation

$$a\partial_x u + b\partial_y u = Du \cdot (a,b) = 0.$$

This means that u is constant along lines to the direction of (a,b). In other words, the solution only depends on the inner product $(x,y)\cdot(b,-a)$ (observe that $(b,-a)\cdot(a,b) = 0$, see the picture in the lectures). That is, there exists differentiable $f : \mathbb{R} \to \mathbb{R}$ such that

$$u(x) = f((x,y)\cdot(b,-a) = f(bx-ay).$$

Indeed, let us check

$$\begin{aligned}
&\partial_x u(x,y) = bf'(bx-ay), \\
&\partial_y u(x,y) = -af'(bx-ay) \\
\Longrightarrow\ & a\partial_x u(x,y) + b\partial_y(x,y) = abf'(bx-ay) - baf'(bx-ay) = 0.
\end{aligned}$$

Example:

$$\begin{cases} 4\partial_x u - 3\partial_y u = 0 \\ u(0,y) = y^3. \end{cases}$$

From the general solution we have

$$u(x,y) = f(-3x-4y).$$

So by the initial condition

$$\begin{aligned}
&u(0,y) = y^3 \\
\Longrightarrow\ & f(-3\cdot 0 - 4y) = y^3 \\
\Longrightarrow\ & f(-4y) = y^3 \\
\Longrightarrow\ & f(t) = -t^3/64.
\end{aligned}$$

And so $u(x,y) = -(-3x-4y)^3/64 = (3x+4y)^3/64$.

(b) Method of characteristics: try to find a "characteristic curve" starting at some point (x_0,y_0) :

$$\{(x(s),y(s)) : x(0) = x_0, y(0) = y_0\}$$

such that

$$z(s) := u(x(s),y(s)) \tag{2.1}$$

is easy to solve along that curve using the PDE. We have

$$\begin{aligned}\frac{d}{ds}z(s) &= \frac{d}{ds}u(x(s),y(s))\\ &= Du(x(s),y(s))\cdot(x'(s),y'(s))\\ &= 0,\end{aligned} \tag{2.2}$$

where the last identity holds if $x'(s)=a$ and $y'(s)=b$. Thus we take

$$x(s)=x_0+sa \quad \text{and} \quad y(s)=y_0+sb.$$

Then by (2.2) the function $z(s)$ is constant and so by (2.1) the solution u is constant on the curve

$$s\mapsto(x(s),y(s))=(x_0+sa,y_0+sb)$$

for any choice of (x_0,y_0). Thus again solutions have the form

$$u(x,y)=f(bx-ay).$$

Example (same as above):

$$\begin{cases}(4,-3)\cdot Du=0\\ u(0,y)=y^3.\end{cases}$$

We take $x(s)=4s$ and $y(s)=-3s+y_0$ so that

$$\frac{d}{ds}z(s)=Du(x(s),y(s))\cdot(x'(s),y'(s))=0.$$

We want to solve the value of u at (x,y). By taking $s=x/4$ and $y_0=3x/4+y$, we have $x(s)=x$ and $y(s)=y$. Thus

$$u(x,y)=u(x(s),y(s))=c \quad \text{for all } s. \tag{2.3}$$

In particular, by the initial condition and definition of $x(s)$ and $y(s)$, we have

$$c=u(x(0),y(0))=u(0,y_0)=y_0^3=(3x/4+y)^3. \tag{2.4}$$

Consequently, by (2.3) and (2.4), we obtain $u(x,y)=(3x/4+y)^3=(3x+4y)^3/64$.

2.2. **Nonconstant coefficients.** We consider equations of the type

$$a(x,y)\partial_x u+b(x,y)\partial_y u=(a(x,y),b(x,y))\cdot Du=0.$$

Example 2.1. Consider

$$y\partial_x u(x,y)-x\partial_y u(x,y)=(y,-x)\cdot Du(x,y)=0. \tag{2.5}$$

Again we use the method of characteristics: We try to deduce behavior of u on suitable curves. Consider the curve

$$\Gamma_{x_0,y_0}:=\{(x(s),y(s)):x(0)=x_0,\ y(0)=y_0,\ s\in\mathbb{R}\}$$

starting at $(x_0,y_0)\in\mathbb{R}^2$ and set

$$z(s):=u(x(s),y(s)) \quad \text{for all } s\in\mathbb{R}.$$

We let $x(s)$ and $y(s)$ solve the ordinary differential equation pair

$$\begin{cases}x'(s)=y(s), & x(0)=x_0,\\ y'(s)=-x(s), & y(0)=y_0,\end{cases} \tag{2.6}$$

so that by (2.5) we have

$$\begin{aligned}\frac{d}{ds}z(s) &= (x'(s),y'(s))\cdot Du(x(s),y(s))\\ &= (y(s),-x(s))\cdot Du(x(s),y(s)) = 0 \quad \text{for all } s\in\mathbb{R}.\end{aligned}$$

This, by definition of $z(s)$, means that u is constant along the curve Γ_{x_0,y_0}. On the other hand, Γ_{x_0,y_0} depends on the equation pair (2.6) which can be solved using the means of the course "differential equations". Differentiating the first equation we get

$$x''(s) = y'(s) = -x(s)$$

and thus the pair (2.6) is solved by

$$\begin{cases} x(s) = c_1\cos(s)+c_2\sin(s),\\ y(s) = -c_1\sin(s)+c_2\cos(s), \end{cases} \qquad \text{whenever } c_1,c_2\in\mathbb{R}.$$

Observe that since $(x(0),y(0)) = (x_0,y_0)$, we have $c_1 = x_0$, $c_2 = y_0$ and that the above is just the equation of circle:

$$x^2(s)+y^2(s) = c_1^2(\sin^2(s)+\cos^2(s))+c_2^2(\sin^2(s)+\cos^2(s)) = c_1^2+c_2^2.$$

In other words Γ_{x_0,y_0} is a circle that contains the point (x_0,y_0). Since (x_0,y_0) was arbitrary, we now know that u is constant in any $(0,0)$-centered circle. Thus it can be written in the form

$$u(x,y) = f(|(x,y)|) = f(\sqrt{x^2+y^2}).$$

3. Transport equation

3.1. **Homogeneous.** We consider

$$\begin{cases} \partial_t u + b\cdot Du = 0 & \text{in } \mathbb{R}^N\times(0,\infty)\\ u = g & \text{on } \mathbb{R}^N\times\{t=0\}, \end{cases} \tag{3.1}$$

where

$$\begin{aligned}&u:\mathbb{R}^N\times(0,\infty)\to\mathbb{R} \text{ (unknown)},\\ &g:\mathbb{R}^N\to\mathbb{R},\ g\in C^1 \text{ (given)},\\ &b\in\mathbb{R}^N \text{ (given)},\\ &Du = (\partial_{x_1},\dots,\partial_{x_N}) \text{ (spatial gradient)}.\end{aligned} \tag{3.2}$$

We already gave a rough derivation of (3.1) this at the beginning in 1D.

Let us solve (3.1) using the method of characteristics. We consider the curve

$$\Gamma_{x_0} = \{(\tilde{x}(s),\tilde{t}(s)) : s\in[0,\infty)\}$$

starting from $(x_0,0)$, where

$$\begin{cases} \tilde{x}(s) = bs + x_0,\\ \tilde{t}(s) = s. \end{cases}$$

Then we have

$$\begin{aligned}\frac{d}{ds}(u(\tilde{x}(s),\tilde{t}(s))) &= Du(\tilde{x}(s),\tilde{t}(s))\cdot\frac{d}{ds}\tilde{x}(s)+\partial_t u(\tilde{x}(s),\tilde{t}(s))\frac{d}{ds}\tilde{t}(s)\\ &= Du(\tilde{x}(s),\tilde{t}(s))\cdot b+\partial_t u(\tilde{x}(s),\tilde{t}(s)) = 0,\end{aligned}$$

which means that u is constant along the curve Γ_{x_0}. This constant must be $g(x_0)$ since $(x_0,0)\in\Gamma_{x_0}$ and $u(x_0,0) = g(x_0)$ by the initial condition in (3.1). Thus we have $u\equiv g(x_0)$

on Γ_{x_0}. Since an arbitrary point $(x,t) \in \mathbb{R}^N \times [0,\infty)$ is on the curve Γ_{x-bt}, we deduce the identity

$$u(x,t) = g(x-bt).$$

Remark 3.1. In order for u to be a classical solution, we require $g \in C^1$. However, even if the original mass distribution is rough, still $g(x-bt)$ seems to make sense as a solution. This suggests the need for a weaker concept of solution.

3.2. **Inhomogeneous.** We consider

$$\begin{cases} \partial_t u + b \cdot Du = f(x,t) & \text{in } \mathbb{R}^N \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^N \times \{t=0\}, \end{cases} \tag{3.3}$$

where

$$f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R},$$

and the rest of the quantities are as in (3.2).

Example 3.2. We continue Example 1.2 but now we in addition drop material on the conveyor belt the amount $f(x,t)$ measured in $kg/(ms)$. Thus

$$\int_x^{x+h} u(y,t+s)\,dy - \int_x^{x+h} u(y,t) \approx u(x,t)bs - u(x+h,t)bs + s\int_x^{x+h} f(y,t)\,dy$$

$$\implies \frac{1}{h}\int_x^{x+h} \frac{u(y,t+s)-u(y,t)}{s}\,dy \approx \frac{u(x,t)-u(x+h,t)}{h}b + \frac{1}{h}\int_x^{x+h} f(y,t)\,dy.$$

Letting $s \to 0$, $h \to 0$, we get

$$\partial_t u - b\partial_x u = f$$

and f represents a source (or a sink if negative).

Let us solve (3.3) using the method of characteristics. We consider the curve

$$\Gamma_{x_0} = \{(\tilde{x}(s),\tilde{t}(s)) : s \in [0,\infty)\}$$

starting from $(x_0,0)$, where

$$\begin{cases} \tilde{x}(s) = bs + x_0, \\ \tilde{t}(s) = s. \end{cases}$$

Then we have

$$\begin{aligned} \frac{d}{ds}(u(\tilde{x}(s),\tilde{t}(s))) &= Du(\tilde{x}(s)) \cdot \frac{d}{ds}\tilde{x}(s) + \partial_t u(\tilde{t}(s))\frac{d}{ds}\tilde{t}(s) \\ &= Du(\tilde{x}(s)) \cdot b + \partial_t u(\tilde{t}(s)) = f(\tilde{x}(s),\tilde{t}(s)). \end{aligned}$$

It follows by the fundamental lemma of calculus that

$$u(\tilde{x}(s),\tilde{t}(s)) = u(\tilde{x}(0),\tilde{t}(0)) + \int_0^s f(\tilde{x}(r),\tilde{t}(r))\,dr \quad \text{for all } s \in [0,\infty). \tag{3.4}$$

Given an arbitrary point $(x,t) \in \mathbb{R}^N \times [0,\infty)$, we set $x_0 = x - bt$. Then we have $(\tilde{x}(t),\tilde{t}(t)) = (x,t)$ which means that $(x,t) \in \Gamma_{x_0}$. Thus by (3.4) we obtain

$$\begin{aligned} u(x,t) = u(\tilde{x}(t),\tilde{t}(t)) &= u(\tilde{x}(0),\tilde{t}(0)) + \int_0^t f(\tilde{x}(r),\tilde{t}(r))\,dr \\ &= u(x-bt,0) + \int_0^t f(br+x-bt,r)\,dr \\ &= g(x-bt) + \int_0^t f(b(r-t)+x,r)\,dr. \end{aligned}$$

Example 3.3. Suppose that there is a decay of mass (for some weird reason) comparable to the amount of mass and time by factor c, no source. Then

$$\int_x^{x+h} u(y,t+s)\,dy - \int_x^{x+h} u(y,t) \approx u(x,t)bs - u(x+h,t)bs - sc\int_x^{x+h} u(y,t)\,dy$$
$$\implies \frac{1}{h}\int_x^{x+h} \frac{u(y,t+s)-u(y,t)}{s}\,dy \approx \frac{u(x,t)-u(x+h,t)}{h}b + \frac{1}{h}\int_x^{x+h} cu(y,t)\,dy.$$

This gives an equation

$$\partial_t u + b\partial_x u + cu = 0.$$

4. Laplace equation

We consider the Laplace equation

$$\Delta u = 0,$$

where $\Delta u = \sum_{i=1}^N D_{ii}u = \sum_{i=1}^N \partial_{x_i}\partial_{x_i}u$. We also consider the Poisson equation

$$-\Delta u = f.$$

Definition 4.1. Solutions $u \in C^2(\Omega)$ to the Laplace equation $\Delta u = 0$ are called *harmonic*.

Usually we consider an open set $\Omega \subset \mathbb{R}^N$ and given boundary values $g : \partial\Omega \to \mathbb{R}$, $g \in C(\partial\Omega)$ and look for the solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Example 4.2. (Equilibrium of diffusion) The Laplace equation models the equilibrium of diffusion. Let $U \subset \Omega$ be a smooth subset and consider the net flux through the boundary ∂U:

$$0 \overset{equilibrium}{=} \int_{\partial U} F\cdot \nu\,dS \overset{div-thm}{=} \int_U \operatorname{div}(F)\,dx,$$

where ν is the exterior unit normal vector. If this holds for every $U \subset \Omega$, it is reasonable to assert that

$$\operatorname{div}(F) = 0.$$

Think for example heat transfer, it is reasonable to assert that flux depends on the difference: heat flows from hot to cold, and faster the greater the difference. Thus we set

$$F = -aDu$$

and get

$$0 = \operatorname{div}(-aDu) = -a\Delta u.$$

Next, suppose that there is a heat source/sink $f : \Omega \to \mathbb{R}$. Then the net flux equals $\int_U f\,dx$

$$\int_U f\,dx = \int_{\partial U} F\cdot \nu dS = \int_U \operatorname{div}(F)\,dx$$

and with $F = -Du$ we get the Poisson equation

$$-\Delta u = f.$$

The boundary values $u = g$ on $\partial\Omega$ model a situation where temperatures, voltages or chemical concentrations are given/known at the boundary and we try to find them inside.

Example 4.3. Let $\Omega = (0,1)$ and consider

$$\begin{cases} \Delta u(x) = u''(x) = 0 & \text{in } \Omega \\ u(0) = 0, u(1) = 2. \end{cases}$$

Then $u(x) = ax + b$ and from

$$u(0) = b = 0, \quad u(1) = a1 = 2,$$

so that the solution is $u(x) = 2x$. Without the unique boundary values we of course couldn't have found the unique solution. This is natural also from the point of view of physical applications above.

Consider then the Poisson equation

$$\begin{cases} -\Delta u = -u'' = 1 & \text{in } \Omega \\ u(0) = 0, u(1) = 0. \end{cases}$$

Then $u'(x) = -x + a$ and $u(x) = -\frac{1}{2}x^2 + ax + b$ and

$$u(0) = b = 0, \quad u(1) = -\frac{1}{2} + a = 0$$

so that the solution is $u(x) = -\frac{1}{2}x^2 + \frac{1}{2}x$.

4.1. **Fundamental solution.** We seek a solution $u : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ to Laplace's equation that is radially symmetric:

$$u(x) = v(|x|)$$

for some $v : (0,\infty) \to \mathbb{R}$. We denote $r(x) := |x|$ so that $u(x) = v(r(x))$. Then

$$\begin{aligned} r &= (\sum_{i=1}^{N} x_i^2)^{1/2}, \\ D_i r &= \frac{2x_i}{2(\sum_{i=1}^{N} x_i^2)^{1/2}} = \frac{x_i}{r}, \\ D_{ii} r &= \frac{1}{r} - x_i \frac{D_i r}{r^2} = \frac{1}{r} - \frac{x_i^2}{r^3}. \end{aligned}$$

Thus by chain rule

$$\begin{aligned} D_i u &= D_i v(r) = v'(r) D_i r = v'(r) \frac{x_i}{r}, \\ D_{ii} u &= D_i(v'(r) D_i r) = v''(r) \frac{x_i}{r} D_i r + v'(r) D_{ii} r = v''(r) \frac{x_i^2}{r^2} + v'(r)(\frac{1}{r} - \frac{x_i^2}{r^3}). \end{aligned}$$

Therefore

$$\begin{aligned} 0 = \Delta u = \sum_{i=1}^{N} D_{ii} u &= \sum_{i=1}^{N} \left(v''(r) \frac{x_i^2}{r^2} + v'(r)(\frac{1}{r} - \frac{x_i^2}{r^3}) \right) \quad \Big| \sum_{i=1}^{N} x_i^2 = r^2 \\ &= v''(r) \frac{r^2}{r^2} + v'(r)(\frac{N}{r} - \frac{r^2}{r^3}) \\ &= v''(r) + v'(r) \frac{N-1}{r}. \end{aligned}$$

Since r was just shorthand for $|x|$ and the above holds for any $x \in \mathbb{R}^N \setminus \{0\}$, we have deduced that

$$v''(s) + v'(s) \frac{N-1}{s} = 0 \quad \text{for all } s \in (0,\infty).$$

Assuming that $v' \neq 0$, we conclude further that

$$(\log(|v'(s)|))' = \frac{v''(s)}{v'(s)} = \frac{1-N}{s} \quad \text{for all } s \in (0,\infty).$$

Denoting $g(s) := \log(|v'(s)|)$, the above reads as

$$g'(s) = \frac{1-N}{s},$$

which is solved by

$$g(s) = a + (1-N)\log s = a + \log s^{1-N} \quad \text{whenever } a \in \mathbb{R}.$$

Thus, v' must satisfy

$$\begin{aligned} &\log(|v'(s)|) = a + \log s^{1-N} \\ \iff\ & e^{\log|v'(s)|} = e^{a+\log s^{1-N}} \\ \iff\ & |v'(s)| = e^a s^{1-N} =: bs^{1-N} \end{aligned} \tag{4.1}$$

for all $s \in (0,\infty)$ and some constant $b > 0$. From (4.1) we conclude that a radial solution to Laplace's equation in $\mathbb{R}^N \setminus \{0\}$ is given by

$$u(x) = v(|x|),$$

where

$$v(s) = \begin{cases} c\log s + d & \text{if } N = 2, \\ cs^{2-N} + d & \text{if } N \geq 3, \end{cases}$$

and $c, d \in \mathbb{R}$ are arbitrary constants. This motivates the following definition.

Definition 4.4 (Fundamental solution)**.** The function

$$\Phi(x) := \begin{cases} c_2 \log(|x|) & \text{if } N = 2, \\ c_N \frac{1}{|x|^{N-2}} & \text{if } N \geq 3, \end{cases}$$

where $c_N \geq 0 \geq c_2$ are explicit constants to be given later in (4.6), is called the fundamental solution to Laplace's equation.

4.2. **Poisson equation.**

Remark 4.5. Observe that the functions

$$\begin{aligned} &x \mapsto \Phi(x) \\ &x \mapsto \Phi(x-y) \\ &x \mapsto \Phi(x-y)f(y) \\ &x \mapsto \Phi(x-y_1)f(y_1) + \Phi(x-y_2)f(y_2) \end{aligned}$$

are harmonic in the set where they are defined. However,

$$x \mapsto \int_{\mathbb{R}^N} \Phi(x-y)f(y)\,dy$$

is not harmonic, even if one might be tempted to calculate

$$\Delta_x \int_{\mathbb{R}^N} \Phi(x-y)f(y)\,dy \stackrel{?}{=} \int_{\mathbb{R}^N} \Delta_x \Phi(x-y)f(y)\,dy.$$

This is one of those cases where the order of the integral and the differential operator Δ_x cannot be changed i.e. $\stackrel{?}{=}$ does not hold (see Remark 4.7).

Theorem 4.6. *Let $f \in C_0^2(\mathbb{R}^N)$. Let u be the convolution of Φ and f i.e.*

$$u(x) := (\Phi * f)(x) := \int_{\mathbb{R}^N} \Phi(x-y) f(y)\, dy.$$

Then

(1) $u \in C^2(\mathbb{R}^N)$,
(2) $-\Delta u = f$ *in* $\mathbb{R}^N$.

Proof. (1) By change of variables

$$u(x) = \int_{\mathbb{R}^N} \Phi(x-y) f(y)\, dy = \int_{\mathbb{R}^N} \Phi(y) f(x-y)\, dy. \tag{4.2}$$

Let $e_i = (0, \ldots, \underbrace{1}_{i \text{ th}}, 0, \ldots 0)$, $h > 0$. First we compute using (4.2)

$$\begin{aligned} \partial_{x_i} u(x) &= \lim_{h \to 0} \frac{u(x+he_i) - u(x)}{h} \\ &= \lim_{h \to 0} \int_{\mathbb{R}^N} \Phi(y) \frac{f(x+he_i-y) - f(x-y)}{h}\, dy \\ &= \int_{\mathbb{R}^N} \Phi(y) \partial_{x_i} f(x-y)\, dy, \end{aligned} \tag{4.3}$$

where the last identity requires justification. Since $f \in C_0^2(\mathbb{R}^N)$, the function $\partial_{x_i} f$ is uniformly continuous i.e. there exists a continuous, increasing function $\omega : [0,\infty) \to [0,\infty)$, $\omega(0) = 0$ such that

$$|\partial_{x_i} f(z_1) - \partial_{x_i} f(z_2)| \le \omega(|z_1 - z_2|) \quad \text{for all } z_1, z_2 \in \mathbb{R}^N.$$

Thus by the mean value theorem

$$\begin{aligned} &\left| \frac{f(x+he_i-y) - f(x-y)}{h} - \partial_{x_i} f(x-y) \right| \quad \left| \begin{array}{l} \text{by m.v. thm. } \exists \xi \in [x-y, x+he_i-y] \text{ s.t.} \\ \quad \frac{f(x+he_i-y)-f(x-y)}{h} = \partial_{x_i} f(\xi) \end{array} \right. \\ &= |\partial_{x_i} f(\xi) - \partial_{x_i} f(x-y)| \\ &\le \omega(|\xi - (x-y)|) \\ &\le \omega(h). \end{aligned}$$

Thus, since $\operatorname{spt} f \subset B_R$ for some $R > 0$, we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} \Phi(y) \left(\frac{f(x+he_i-y) - f(x-y)}{h} - \partial_{x_i} f(x-y) \right) dy \right| \\ &\le \int_{B_R} |\Phi(y)|\, \omega(h)\, dy = \underbrace{\omega(h)}_{\to 0} \underbrace{\int_{B_R} |\Phi(y)|\, dy}_{<\infty} \to 0 \quad \text{as} \quad h \to 0, \end{aligned}$$

which justifies the last identity in (4.3). As the expression at the right hand-side of (4.3) is continuous in x, we conclude that $u \in C^1(\mathbb{R}^N)$. Similarly we could show that

$$\partial_{x_i} \partial_{x_j} u(x) = \int_{\mathbb{R}^N} \Phi(y) \partial_{x_i} \partial_{x_j} f(x-y)\, dy \tag{4.4}$$

and $\partial_{x_i} \partial_{x_j} u(x) \in C(\mathbb{R}^N)$.

(2) Fix $\varepsilon > 0$. By (4.4) we have

$$\begin{aligned}\Delta u(x) &= \int_{B(0,\varepsilon)} \Phi(y)\Delta f(x-y)\,dy + \int_{\mathbb{R}^N\setminus B(0,\varepsilon)} \Phi(y)\Delta f(x-y)\,dy\\ &=: I_\varepsilon + J_\varepsilon.\end{aligned}$$

Observe then that if $N \geq 3$, we have

$$\begin{aligned}\int_{B(0,\varepsilon)} |y|^{2-N} &= \int_0^\varepsilon \int_{\partial B(0,r)} r^{2-N}\,dS\,dr \quad \Big|\, |\partial B(0,r)| = Cr^{N-1}\\ &= C\int_0^\varepsilon r\,dr = C\frac{1}{2}\varepsilon^2\end{aligned}$$

and if $N = 2$

$$\begin{aligned}\int_{B(0,\varepsilon)} -\log(|y|)\,dy &= -\int_0^\varepsilon \int_{\partial B(0,r)} \log r\,dS\,dr\\ &= -C\int_0^\varepsilon r\log r\,dr \qquad \Big|\text{integration by parts}\\ &= C(-\frac{1}{2}\varepsilon^2\log\varepsilon + \frac{1}{4}\varepsilon^2) \leq C\varepsilon^2\,|\log\varepsilon|\,.\end{aligned}$$

Thus

$$\begin{aligned}|I_\varepsilon| = \left|\int_{B(0,\varepsilon)} \Phi(y)\Delta f(x-y)\,dy\right| &\leq \max_{y\in\mathbb{R}^N} |\Delta f(x-y)| \int_{B(0,\varepsilon)} \Phi(y)\,dy\\ &\leq C\begin{cases}\varepsilon^2\,|\log\varepsilon|\,, & N=2,\\ \varepsilon^2, & N\geq 3,\end{cases}\\ &\to 0 \quad \text{as} \quad \varepsilon \to 0.\end{aligned}$$

For J_ε choosing $R > 0$ large enough so that $\operatorname{spt} f \subset B(0,R)$, we can integrate by parts (or use Gauss-Green theorem to be precise) to obtain

$$\begin{aligned}J_\varepsilon &= \int_{B(0,R)\setminus B(0,\varepsilon)} \Phi(y)\Delta f(x-y)\,dy\\ &= -\int_{B(0,R)} D\Phi(y)\cdot Df(x-y)\,dy + \int_{\partial B(0,\varepsilon)} \Phi(y)Df(x-y)\cdot \nu\, dS(y)\\ &=: K_\varepsilon + L_\varepsilon,\end{aligned}$$

where $\nu = \nu(y) = -y/|y|$ is the outwards pointing unit normal vector. Then

$$\begin{aligned}|L_\varepsilon| &= \left|\int_{\partial B(0,\varepsilon)} \Phi(y)\cdot Df(x-y)\cdot \nu\,dS(y)\right|\\ &= C\varepsilon^{N-1} \max_{y\in\partial B(0,\varepsilon)} |\Phi(y)| \max_{y\in\partial B(0,\varepsilon)} |Df(x-y)|\\ &\leq C\varepsilon^{N-1}\begin{cases}|\log\varepsilon|\,, & N=2,\\ \varepsilon^{2-N}, & N\geq 3,\end{cases}\\ &\to 0 \quad \text{as} \quad \varepsilon\to 0.\end{aligned}$$

We integrate K_ε by parts

$$\begin{aligned}
K_\varepsilon &= -\int_{B(0,R)\setminus B(0,\varepsilon)} D\Phi(y)\cdot Df(x-y)\,dy \\
&= \int_{B(0,R)\setminus B(0,\varepsilon)} \underbrace{\operatorname{div} D\Phi(y)}_{=\Delta\Phi=0} f(x-y)\,dy - \int_{\partial B(0,\varepsilon)} D\Phi(y)\cdot\nu f(x-y)\,dS(y) \\
&= -\int_{\partial B(0,\varepsilon)} D\Phi(y)\cdot\nu f(x-y)\,dS(y) \\
&= -\int_{\partial B(0,\varepsilon)} \begin{cases} c_2\frac{y}{|y|^2} & N\geq 2, \\ c_N(2-N)y\,|y|^{-N} & N\geq 3, \end{cases} \cdot\left(-\frac{y}{|y|}\right) f(x-y)\,dS(y) \qquad (4.5)\\
&= \int_{\partial B(0,\varepsilon)} \begin{cases} c_2\,|y|^{-1} & N\geq 2, \\ c_N(2-N)\,|y|^{1-N} & N\geq 3, \end{cases} f(x-y)\,dS(y) \\
&\overset{\text{choose } c_N}{=} -\frac{1}{|\partial B(0,\varepsilon)|}\int_{\partial B(0,\varepsilon)} f(x-y)\,dS(y) \overset{\text{avg.}}{\to} -f(x)
\end{aligned}$$

as $\varepsilon\to 0$. Observe that above $\nu = -\frac{y}{|y|}$ is the exterior normal to $B(0,R)\setminus B(0,\varepsilon)$ on $\partial B(0,\varepsilon)$. Above we had $|y|=\varepsilon$ and fixed c_N so that

$$-\frac{1}{|\partial B(0,\varepsilon)|} = \begin{cases} c_2\varepsilon^{-1}, & N=2 \\ c_N(2-N)\varepsilon^{1-N}, & N\geq 3 \end{cases}$$

and since $|\partial B(0,\varepsilon)| = N\alpha_N\varepsilon^{N-1}$, we get

$$\begin{aligned}
c_2 &= -\frac{1}{2\alpha_2} = -\frac{1}{2\pi}, \\
c_N &= \frac{1}{(N-2)N\alpha_N}, \quad N\geq 3. \qquad (4.6)
\end{aligned}$$

We have thus proven that

$$\Delta u(x) = I_\varepsilon + J_\varepsilon = I_\varepsilon + K_\varepsilon + L_\varepsilon \to -f(x)$$

as $\varepsilon\to 0$. □

Remark 4.7. The Dirac delta distribution denoted by δ_x is defined by the identity

$$\delta_x(f) = f(x)$$

for all $f\in C_0^\infty(\mathbb{R}^N)$. By Theorem 4.6 we thus have

$$-\Delta u(x) = -\Delta\int_{\mathbb{R}^N}\Phi(x-y)f(y)\,dy = \delta_x(f).$$

One can also think δ_x as a measure that satisfies

$$\int_{\mathbb{R}^N} f(y)\,d\delta_x(y) = f(x)$$

so that

$$-\Delta\int_{\mathbb{R}^N}\Phi(x-y)f(y)\,dy = \int_{\mathbb{R}^N} f(y)\,d\delta_x(y).$$

More on this on the course “Measure and integration”.

The proof of Theorem 4.6 also motivates the introduction of the following important tool.

4.3. **Convolution, mollifiers, and approximations.** We denote

$$\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \Omega) > \varepsilon\}$$

which is an open set by continuity of $\operatorname{dist}(x, \partial\Omega)$.

Definition 4.8 (Standard mollifier)**.** Let

$$\eta : \mathbb{R}^N \to \mathbb{R}, \quad \eta(x) = \begin{cases} Ce^{1/(|x|^2-1)}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where C is chosen so that

$$\int_{\mathbb{R}^N} \eta \, dx = 1.$$

Then we set for $\varepsilon > 0$

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^N} \eta(\frac{x}{\varepsilon})$$

which is called the standard mollifier.

Remark 4.9. Observe that

$$\eta_\varepsilon \in C_0^\infty(\mathbb{R}^N), \quad \operatorname{spt} \eta_\varepsilon \subset \overline{B}(0, \varepsilon)$$

and

$$\int_{\mathbb{R}^N} \eta_\varepsilon(x) \, dx = 1.$$

Definition 4.10 (Standard mollification)**.** Let $f : \Omega \to [-\infty, \infty]$, $f \in C(\Omega)$. Then we define standard mollification for f by

$$f_\varepsilon : \Omega_\varepsilon \to \mathbb{R}, \quad f_\varepsilon := \eta_\varepsilon * f,$$

where $\eta_\varepsilon * f(x) = \int_\Omega \eta_\varepsilon(x-y) f(y) \, dy$ denotes the convolution for $x \in \Omega_\varepsilon$.

Definition 4.11. We say that $u_j \to u$ locally uniformly in Ω if $u_j \to u$ uniformly in K for every $K \Subset \Omega$.

Theorem 4.12. *The standard mollification has the following properties for $f \in C(\Omega)$.*

(1)
$$D^\alpha f_\varepsilon = f * D^\alpha \eta_\varepsilon \quad \text{in } \Omega_\varepsilon$$
and
$$f_\varepsilon \in C^\infty(\Omega_\varepsilon).$$

(2) *If $f \in C(\Omega)$, then*
$$f_\varepsilon \to f \text{ locally uniformly in } \Omega.$$

(3) *If $\Omega'' \Subset \Omega' \Subset \Omega$, then*
$$\max_{\Omega''} |f_\varepsilon| \leq \max_{\Omega'} |f|$$
for small enough $\varepsilon > 0$.

Proof. The formula $D^\alpha f_\varepsilon = f * D^\alpha \eta_\varepsilon$ follows using similar techniques as in the proof of Theorem (4.6). Then since $\eta_\varepsilon \in C_0^\infty$ by a direction calculation $f_\varepsilon \in C^\infty$ too. The detailed proof is given in the course PDE2. □

4.4. **Mean value property.** A harmonic function has a remarkable property called the mean value formula: the value at one point (!) determines the average of the function over a ball. It also heuristically connects harmonic functions to the stochastic process called Brownian motion and thus to stock prices, option pricing, etc. It is also a key to many interesting mathematical properties.

Remember that $\fint_A f\,dx = \frac{1}{A}\int_A f\,dx$ denotes the integral average.

Theorem 4.13 (Mean value property = mvp)**.** *Let $u \in C^2(\Omega)$. Then the following are equivalent*

(1) *u is harmonic in Ω*
(2) *Whenever $B(x,r) \Subset \Omega$, we have*

$$u(x) = \fint_{\partial B(x,r)} u(y)\,dS(y) = \fint_{B(x,r)} u(y)\,dy.$$

Proof. "(1) $\implies$ (2) Idea: Set

$$\phi(r) = \fint_{\partial B(x,r)} u(y)\,dS(y).$$

and show that $\phi'(r) = 0$.

To this end, let $z \in B(0,1)$ and perform the change of variables $y = rz + x$ so that $dS(y) = r^{N-1}\,dS(z)$ and

$$\begin{aligned}\phi(r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\,dS(y) &= \frac{|\partial B(0,1)|}{|\partial B(x,r)|} r^{N-1} \fint_{\partial B(0,1)} u(rz+x)\,dS(z)\\ &= \fint_{\partial B(0,1)} u(rz+x)\,dS(z).\end{aligned}$$

Then changing the variables back we obtain

$$\begin{aligned}\phi'(r) &= \fint_{\partial B(0,1)} Du(rz+x)\cdot z\,dS(z) && \Big| z = \frac{y-x}{r}, dS(z) = r^{1-N} dS(y)\\ &= \frac{|\partial B(x,r)|}{|\partial B(0,1)|}\fint_{\partial B(x,r)} Du(y)\cdot\frac{y-x}{r} r^{1-N}\,dS(y)\\ &= \fint_{\partial B(x,r)} Du(y)\cdot\frac{y-x}{r}\,dS(y) && \Big|\text{divergence theorem}\\ &= \frac{1}{|\partial B(x,r)|}\int_{B(x,r)} \operatorname{div} Du(y)\,dy && \Big|\operatorname{div} Du = \Delta u = 0\\ &= 0,\end{aligned}$$

where we used that $\frac{y-x}{r}$ is the exterior unit normal vector. Since $\phi'(r) = 0$, ϕ has a constant value that has to be

$$\lim_{r\to 0} \fint_{\partial B(x,r)} u(y)\,dy = u(x).$$

Moreover

$$\begin{aligned}
\fint_{B(x,r)} u(y)\,dy &= \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\,dy \\
&= \frac{1}{|B(x,r)|}\int_0^r \int_{\partial B(x,s)} u\,dS\,ds \\
&= \frac{1}{|B(x,r)|}\int_0^r |\partial B(x,s)| \fint_{\partial B(x,s)} u\,dS\,ds \\
&= \frac{1}{|B(x,r)|}\int_0^r |\partial B(x,s)|\, u(x)\,ds \\
&= u(x).
\end{aligned}$$

"(1) $\Longleftarrow$ (2)" Assume thriving for a contradiction that u is not harmonic even if the mean value property holds i.e. that there is x so that $\Delta u(x) > 0$ and by continuity even in a small ball around x. Then using the above calculation

$$\phi'(r) = \frac{1}{|\partial B(x,r)|}\fint_{\partial B(x,r)} \Delta u(y) > 0$$

so that the mean value cannot be a constant, a contradiction. $\square$

Example 4.14. Let $\Omega = (0,1)$ and $\Delta u = u'' = 0$. Then $u(y) = ay + b$ and

$$\frac{1}{2r}\int_{x-r}^{x+r} u(y)\,dy = \frac{1}{2r}\int_{x-r}^{x+r} (ay+b)\,dr = ax + b.$$

4.5. Properties of harmonic functions.

Example 4.15. Let $\Omega = (0,1)$ and $\Delta u = u'' = 0$. Then $u(y) = ay + b$. In particular, u obtains its largest and smallest values at the boundary. This also holds in higher dimensions as seen in the next theorem.

Theorem 4.16 (Max principles)**.** *Let Ω be a bounded open set and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ a harmonic function in Ω. Then*

(1) *(weak max principle)* $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$*,*
(2) *(strong max principle) if Ω is connected and there is $x_0 \in \Omega$ such that*

$$u(x_0) = \max_{\overline{\Omega}} u, \tag{4.7}$$

then u is constant in Ω.

Proof. (2) Suppose that (4.7) holds. Then for $r > 0$ such that $B(x_0, r) \Subset \Omega$ we have

$$M := u(x_0) \overset{\text{mvp}}{=} \fint_{B(x_0,r)} u\,dy$$

by the mean value property. Since M is the maximum of u in $\overline{\Omega}$, the average on the right can equal M only if

$$u \equiv M \quad \text{in } B(x_0, r).$$

Since Ω is connected, we deduce that

$$u \equiv M \quad \text{in } \Omega.$$

(1) Suppose on the contrary that $\max_{\overline{\Omega}} u > \max_{\partial\Omega} u$. Then there is a maximum point inside the domain and by the strong maximum principle this is a contradiction.

Second proof: For later we use we also give a proof that does not use the strong maximum principle. Assume without loss of generaility that $\Omega = B(0,1)$ and $\max_{\overline{\Omega}} u >$

$\max_{\partial\Omega} u + 2\varepsilon$ for some $\varepsilon > 0$. Then $v(x) = u(x) + \varepsilon |x|^2/2$ also attains a maximum at some point $z_0 \in \Omega$. At the max point it holds that

$$\Delta v(z_0) \le 0$$

but on the other hand $\Delta v = \Delta u + \varepsilon N = 0 + \varepsilon N > 0$, a contradiction. □

Remark 4.17. Also $-u$ is harmonic and thus we obtain a minimum principle.

Remark 4.18. Obviously the mean value principle or maximum principle does not hold for the Poisson equation. Recall that for

$$\begin{cases} -\Delta u = -u''(x) = 1 & \text{in } \Omega = (0,1) \\ u(0) = 0, u(1) = 0 \end{cases}$$

we have $u(x) = -\frac{1}{2}x^2 + \frac{1}{2}x$.

Theorem 4.19 (Uniqueness to Dirichlet problem)**.** *Let Ω be a bounded open set. Let $g \in C(\partial\Omega)$ and $f \in C(\Omega)$. Then the problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

has at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Proof. Let u and v be two solutions. Then $w = u - v$ solves

$$\Delta w = \Delta(u - v) = f - f = 0$$

with boundary values $w = 0$. By the weak maximum principle

$$w = u - v \le 0 \quad \text{in } \Omega.$$

By setting, $w = v - u$ we also get

$$v - u \le 0 \quad \text{in } \Omega.$$

□

Mean value property has also other perhaps surprising consequences.

Theorem 4.20 (Smoothness)**.** *If $u \in C(\Omega)$ satisfies the mean value property*

$$u(x) = \fint_{\partial B(x,r)} u \, dS$$

for every $B(x_0, r) \Subset \Omega$. Then

$$u \in C^\infty(\Omega).$$

In particular, harmonic functions are smooth.

Proof. Fix $\varepsilon > 0$. Denote by $u_\varepsilon = \eta_\varepsilon * u$ the mollification by convolution as in Definition 4.10 and $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. By Theorem 4.12 we have $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$. The claim then follows from the observation that $u = u_\varepsilon$ when the mean value property holds. Indeed, let $x \in \Omega_\varepsilon$ and compute

$$
\begin{aligned}
u_\varepsilon(x) = \eta_\varepsilon * u = \int_{\mathbb{R}^N} \eta_\varepsilon(x-y)u(y)\,dy \\
&= \frac{1}{\varepsilon^N}\int_{B(x,\varepsilon)} \eta(\frac{x-y}{\varepsilon})u(y)\,dy && \Big|\eta \text{ radial} \\
&= \frac{1}{\varepsilon^N}\int_0^\varepsilon \eta(\frac{re_1}{\varepsilon})\int_{\partial B(x,r)} u(y)\,dS(y)\,dr && \Big|\text{mvp} \\
&= \frac{u(x)}{\varepsilon^N}\int_0^\varepsilon \eta(\frac{re_1}{\varepsilon})\,|\partial B(x,r)|\,dr \\
&= \frac{u(x)}{\varepsilon^N}\int_0^\varepsilon \eta(\frac{re_1}{\varepsilon})\omega_N r^{N-1}\,dr && \Big|\eta_\varepsilon(x) = \frac{1}{\varepsilon^N}\eta(\frac{x}{\varepsilon}) \\
&= u(x)\int_{B(x,\varepsilon)} \eta_\varepsilon(y)\,dy \\
&= u(x).
\end{aligned}
$$

□

My mvp, harmonic function satisfies

$$|u(x)| \le \fint_B |u|\,dy.$$

Also the derivatives have integral estimates.

Theorem 4.21 (Derivative estimates)**.** *Let u be harmonic in Ω. Then*

$$
\begin{aligned}
|D_i u(x_0)| &\le \frac{c_1}{r^{N+1}}\int_{B(x_0,r)} |u|\,dy, \\
\left|D_{ij} u(x_0)\right| &\le \frac{c_2}{r^{N+2}}\int_{B(x_0,r)} |u|\,dy,
\end{aligned}
$$

where $c_i = c(N,i)$ for $B(x_0,r) \Subset \Omega$.

Proof. The idea is to differentiate under the integral. If we can justify $*$ below, we get

$$
\begin{aligned}
|D_i u(x_0)| &\overset{*}{=} \left|\fint_{B(x_0,r/2)} D_i u(y)\,dy\right| && \Big|\text{Gauss-Green} \\
&= \frac{|\partial B(x_0,r/2)|}{|B(x_0,r/2)|}\left|\fint_{\partial B(x_0,r/2)} u\nu_i\,dS\right| \\
&\le \frac{c}{r}\max_{\partial B(x_0,r/2)} |u|\,.
\end{aligned}
$$

Let $x \in \partial B(x_0,r/2)$ and observe that $B(x,r/2) \subset B(x_0,r)$ so that

$$
\begin{aligned}
|u(x)| &= \left|\frac{1}{|B(x,r/2)|}\int_{B(x,r/2)} u\,dy\right| \\
&\le \frac{|B(x,r)|}{|B(x,r/2)|}\fint_{B(x_0,r)} |u|\,dy \\
&= c\fint_{B(x_0,r)} |u|\,dy.
\end{aligned}
$$

Combining the two estimates yields the result if we can justify $*$. To show $*$, observe

$$D_i u(x_0) = \frac{\partial}{\partial_{x_{0_i}}} \fint_{B(0,r/2)} u(x_0+y)\,dy = \fint_{B(0,r/2)} D_i u(x_0+y)\,dy$$

where the last equality follows from the fact that for any $\varepsilon > 0$ there is $h > 0$ such that

$$\left| \int_{B(0,r/2)} \partial_i u(x_0+y+he_i) - \frac{u(x_0+y+he_i)-u(x_0+y)}{h}\,dy \right| \le \varepsilon$$

as in the proof of Theorem 4.6. Another alternative is to observe that $D_i\Delta u = \Delta D_i u = 0$ so that also the partial derivatives satisfy the mean value property in $*$.

Then for the second derivatives we get similarly as above by slightly adjusting radii

$$\begin{aligned}
|D_{ij}u| &= \left| \fint_{B(x_0,r/2)} D_{ij}u\,dx \right| \\
&= \frac{|\partial B(x_0,r/2)|}{|B(x_0,r/2)|} \left| \fint_{B(x_0,r/2)} D_i u \nu_j\,dS \right| \\
&\le \frac{c}{r} \max_{\partial B(x_0,r/2)} |D_i u| \le \frac{cc_1}{r^{N+2}} \int_{B(x_0,r)} |u|\,dy.
\end{aligned}$$

□

Example 4.22. Let $\Omega = (0,1)$ and u harmonic, then $u(x) = ax+b$ and let for simplicity $a,b \ge 0$. Observe

$$\begin{aligned}
&|Du_i| = a, \\
&\frac{1}{r^2}\int_{x-r}^{x+r} |u|\,dy = \frac{4axr}{2r^2} + \frac{2b}{r} \ge \frac{2ax}{r} \ge 2a
\end{aligned}$$

assuming $(x-r, x+r) \subset \Omega$.

Observe that the result is independent of the domain. Also observe that by $u(x) = ax+b$ immediately tells that it is natural to have dependence of the size of u on the right hand side.

Next we utilize the observation that the estimate is independent of the domain.

Corollary 4.23 (Liouville theorem)**.** *If u is bounded and harmonic in $\mathbb{R}^N$, then u is a constant.*

Proof. Since there is a constant $M \ge 0$ such that $|u| \le M$, by the previous theorem

$$|D_i u(x)| \le \frac{c}{r^{N+1}} \int_{B(x,r)} |u|\,dy \le \frac{c}{r} \fint_{B(x,r)} M\,dy = \frac{c_1 M}{r} \to 0$$

as $r \to \infty$, we see that $D_i u = 0$ at every point for any $i = 1,\dots,N$. Thus u is a constant. □

Corollary 4.24 (Uniqueness in $\mathbb{R}^N$)**.** *Let $f \in C_0^2(\mathbb{R}^N)$, $N \ge 3$. Then every bounded solution $u \in C^2(\mathbb{R}^N)$ to*

$$-\Delta u = f,$$

is of the form

$$u(x) = (\Phi * f)(x) + c = \int_{\mathbb{R}^N} \Phi(x-y) f(y)\,dy + c,$$

where c is a constant.

Proof. Let $v(x) = \int_{\mathbb{R}^N} \Phi(x-y) f(y)\,dy$. We have shown that $v \in C^2(\mathbb{R}^N)$ and $-\Delta v = f$. Let $\operatorname{spt} f \subset B(0,r)$. There is M such that $|v| \le M$ in $B(0,2r)$. Let $x \notin B(0,2r)$. Then

$$|v(x)| \le \left| \int_{\mathbb{R}^N} \Phi(\underbrace{x-y}_{|\cdot|>r \text{ if } y \in \operatorname{spt} f}) f(y)\,dy \right| \le c r^{2-N} \int_{\mathbb{R}^N} |f|\,dy \le c r^{2-N}$$

i.e. v is a bounded solution. Let u be another bounded solution. Then

$$\Delta(u-v) = 0$$

and Liouville's theorem implies that $u - v = c$. □

Remark 4.25. Previous theorem is false without boundedness.

Theorem 4.26. *Let u be harmonic in Ω. Then u is real analytic in Ω.*

Proof. (Sketch) We have shown that $u \in C^\infty$ and we want to show that u can be even presented by a convergent power series around a point.

Set

$$R_N(x) := u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)(x-x_0)^\alpha}{\alpha!},$$

where $(x-x_0)^\alpha = (x-x_0)_1^{\alpha_1} \cdots (x-x_N)_N^{\alpha_N}$. By Taylor's theorem

$$R_N(x) = \sum_{|\alpha|=N} \frac{D^\alpha u(x_0 + t(x-x_0))(x-x_0)^\alpha}{\alpha!}$$

for some $t \in [0,1]$. One could establish higher derivative estimate with a sharp coefficient similarly as in Theorem (4.21), and plugging in such an estimate, we see that

$$|R_N(x)| \to 0.$$

□

Theorem 4.27 (Harnack's inequality)**.** *Let $u \ge 0$ be harmonic in Ω and $B(x_0, 4r) \subset \Omega$. Then for $c = 3^N$ it holds that*

$$\sup_{B(x_0,r)} u \le c \inf_{B(x_0,r)} u.$$

Proof. Let $x, y \in B(x_0, r)$, then by mean value property

$$u(y) = \fint_{B(y,3r)} u\,dz \ge \frac{|B(x,r)|}{|B(y,3r)|} \fint_{B(x,r)} u\,dz = \frac{1}{3^N} u(x).$$

The claim follows since $x, y \in B(x_0, r)$ were arbitrary. □

Corollary 4.28 (Harnacks inequality, general form)**.** *Let $u \ge 0$ be harmonic in Ω and $V \Subset \Omega$ be a connected open set. Then there is $c = c(N, V, \Omega) > 0$ such that*

$$\sup_V u \le c \inf_V u.$$

Proof. Idea: covering argument. Let $r = \operatorname{dist}(\overline{V}, \partial\Omega)/4$,

$$\overline{V} \subset \left\{ B(x_\gamma, r) \right\}_\gamma.$$

By compactness, there is a finite subcover

$$\overline{V} \subset \{B(x_i, r)\}_{i=1}^n.$$

Then for $x, y \in V$, use Harnack n times to obtain

$$u(y) \geq (1/3)^{Nn} u(x).$$

Since n depends only on geometry of V and Ω, we have $(1/3)^{Nn} = C(N, V, \Omega)$. □

Remark 4.29. The assumption $u \geq 0$ is essential: let $\Omega = (-1, 1)$ and $u(x) = x$.

Harnack's inequality implies strong maximum principle. We have already proved the maximum principle using the mean value property, but for many equations Harnack's inequality holds but mean value property does not.

Corollary 4.30 (Strong max principle)**.** *Let Ω be a bounded, open and connected set and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ harmonic in Ω. Then, if there is $x_0 \in \Omega$ such that*

$$u(x_0) = \max_{\overline{\Omega}} u$$

it follows that

$$u \text{ is constant in } \Omega.$$

Proof. Let $M = u(x_0) = \max_{\overline{\Omega}} u$ and set

$$v := M - u.a$$

Then v is harmonic, $v \geq 0$ and $v(x_0) = 0$. Choose connected $V \ni x_0$ s.t. $V \Subset \Omega$. Then

$$0 \leq \sup_V v \leq C \inf_V v \leq C v(x_0) = 0.$$

□

4.6. **Green's functions.** We are going to look for a so called Green's function that helps to represent the solution to the boundary value Poisson problem.

Theorem 4.31. *Let $\partial\Omega \in C^1$, $u \in C^2(\overline{\Omega})$. If u solves*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

then

$$u(x) = \int_\Omega f(y) G(x,y)\, dy - \int_{\partial\Omega} g(y) \frac{\partial}{\partial \nu} G(x,y)\, dS(y),$$

where G is Green's function.

Remark 4.32. Observe that this resembles Φ since $u = \Phi * f$ solved

$$-\Delta u = f$$

in $\mathbb{R}^N$ under suitable assumptions. Now in addition we have boundary conditions.

The theorem says: if there is such u and we can find G, then we have solved the Poisson problem. However, finding G can be difficult and usually we can derive explicit formulas in the simple domains (like ball, later) only.

Derivation of Green's function. First recall that from Gauss-Green formula we obtain the integration by parts formula

$$\int_U uD_i\varphi\,dy = \int_{\partial U} \varphi u\nu_i\,dS(y) - \int_U \varphi D_i u\,dy. \tag{4.8}$$

If $u,\varphi \in C^2(\overline{U})$, we can use this twice to obtain

$$\begin{aligned}
\int_U u\Delta\varphi\,dy &= \sum_{i=1}^N \int_U uD_i(D_i\varphi)\,dy \\
&= \sum_{i=1}^N \Big(\int_{\partial U} uD_i\varphi\nu_i\,dS(y) - \int_U D_iuD_i\varphi\,dy\Big) \\
&= \sum_{i=1}^N \Big(\int_{\partial U} uD_i\varphi\nu_i\,dS(y) - \int_{\partial U} \varphi D_iu\nu_i + \int_U \varphi D_i(D_iu)\,dy\Big) \\
&= \int_{\partial U} u\frac{\partial\varphi}{\partial\nu}\,dS(y) - \int_{\partial U} \varphi\frac{\partial u}{\partial\nu}\,dS(y) + \int_U \varphi\Delta u,
\end{aligned}$$

which is called *Green's formula*.

Let now Ω be bounded, $\partial\Omega \in C^1$ and suppose that $u \in C^2(\overline{\Omega})$ solves the Poisson problem. Let $B_\varepsilon(x) \Subset \Omega$ and set

$$\Phi^x(y) := \Phi(y-x) \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\}.$$

From Green's formula we obtain

$$\begin{aligned}
&\int_{\Omega\setminus B(x,\varepsilon)} u\Delta\Phi^x\,dy - \int_{\Omega\setminus B(x,\varepsilon)} \Phi^x\Delta u\,dy \\
&= \int_{\partial(\Omega\setminus B(x,\varepsilon))} u\frac{\partial\Phi^x}{\partial\nu}\,dS(y) - \int_{\partial(\Omega\setminus B(x,\varepsilon))} \Phi^x\frac{\partial u}{\partial\nu}\,dS(y),
\end{aligned} \tag{4.9}$$

where ν is the exterior unit normal vector to $\Omega\setminus B_\varepsilon(x)$ i.e. it points towards x on $\partial B(x,\varepsilon)$ and outwards from Ω on $\partial\Omega$. Thus we have

$$\begin{aligned}
\int_{\partial B(x,\varepsilon)} u\frac{\partial\Phi^x}{\partial\nu}\,dS(y) &= \int_{\partial B(0,\varepsilon)} u(x-y)\frac{\partial\Phi}{\partial\nu}(y)\,dy \qquad \Big|\text{c.f. (4.5)} \\
&= \fint_{\partial B(0,\varepsilon)} u(x)\,dy \to u(x) \quad \text{as } \varepsilon\to 0.
\end{aligned} \tag{4.10}$$

Similarly, using that the derivatives of u are bounded (since $u \in C^2(\overline{\Omega})$), we obtain

$$\left|\int_{\partial B(x,\varepsilon)} \Phi^x\frac{\partial u}{\partial\nu}\,dS(y)\right| \le c\varepsilon^{N-1}\max_{\partial B(x,\varepsilon)}|\Phi^x| \le c\varepsilon^{N-1}\begin{cases}\varepsilon^{N-2}, & N\ge 3,\\ |\log\varepsilon|, & N=2,\end{cases} \to 0 \quad \text{as } \varepsilon\to 0. \tag{4.11}$$

Letting $\varepsilon \to 0$, it follows from (4.10) and (4.11) that the right-hand side of (4.9) converges to

$$u(x) + \int_{\partial\Omega} u\frac{\partial\Phi^x}{\partial\nu}\,dS(y) - \int_{\partial\Omega} \Phi^x\frac{\partial u}{\partial\nu}\,dS(y).$$

On the other-hand, the left-hand side of (4.9) readily converges to

$$\int_\Omega u\overbrace{\Delta\Phi^x}^{=0}\,dy - \int_\Omega \Phi^x\Delta u\,dy = -\int_\Omega \Phi^x\Delta u\,dy.$$

Hence we obtain

$$u(x) = -\int_{\partial\Omega} u \frac{\partial \Phi^x}{\partial \nu}\, dS(y) + \int_{\partial\Omega} \Phi^x \frac{\partial u}{\partial \nu}\, dS(y) - \int_{\Omega} \Phi^x \Delta u\, dy \tag{4.12}$$

for any $x \in \Omega$ and $u \in C^2(\overline{\Omega})$.

Since the formulation of a Poisson problem gives us $-\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$, formula (4.12) almost lets us solve u. However, the middle term is a problem since it is not directly prescribed by f or g. To deal with this term, the idea is to introduce a corrector $\varphi^x = \varphi^x(y)$ that solves the boundary value problem

$$\begin{cases} \Delta \varphi^x = 0 & \text{in } \Omega, \\ \varphi^x = \Phi^x(y) = \Phi(y-x) & \text{on } \partial\Omega. \end{cases} \tag{4.13}$$

Then by Green's formula we see that the middle term in (4.12) satisfies

$$\int_{\partial\Omega} \Phi^x \frac{\partial u}{\partial \nu}\, dS(y) = \int_{\Omega} \varphi^x \Delta u\, dy + \int_{\partial\Omega} u \frac{\partial \varphi^x}{\partial \nu}\, dS(y).$$

Plugging this into (4.12) we obtain

$$\begin{aligned} u(x) &= \int_{\partial\Omega} u\Big(\frac{\partial \varphi^x}{\partial \nu} - \frac{\partial \Phi^x}{\partial \nu}\Big)\, dS(y) + \int_{\Omega} (\varphi^x - \Phi^x)\Delta u\, dy \\ &= \int_{\partial\Omega} g \frac{\partial}{\partial \nu}(\varphi^x - \Phi(y-x))\, dS(y) + \int_{\Omega} (\Phi(y-x) - \varphi^x) f\, dy, \end{aligned} \tag{4.14}$$

where we recalled definition of Φ^x and used that u solves the Poisson problem. Observe that the right-hand side of (4.14) is independent of u and can be calculated provided that Ω is so regular that φ^x can be solved for any $x \in \Omega$. This leads to the following definition.

Definition 4.33. *Green's function* for the region Ω is

$$G(x,y) = \Phi(y-x) - \varphi^x(y), \quad x, y \in \Omega, x \neq y.$$

Remark 4.34.

- Observe that (4.14) can be now written as

$$u(x) = \int_{\Omega} G(x,y) f(y)\, dy - \int_{\partial\Omega} \frac{\partial}{\partial \nu} G(x,y) g(y)\, dS(y).$$

- If

$$\begin{cases} \Delta u = f = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

  we get the *Poisson formula*

$$u(x) = \int_{\partial\Omega} K(x,y) g(y)\, dS(y),$$

  where

$$K(x,y) = -\frac{\partial G(x,y)}{\partial \nu}$$

  is called the *Poisson kernel*.
- Formally $-\Delta_y G(x,y) = -\Delta_y(\Phi(y-x) - \varphi^x(y)) = \delta_x - 0$ in Ω, and $G(x,y) = \Phi(y-x) - \Phi(y-x) = 0$ for $y \in \partial\Omega$, i.e.

$$\begin{cases} -\Delta_y G(x,y) = \delta_x, & y \in \Omega, \\ G(x,y) = 0, & y \in \partial\Omega. \end{cases}$$

4.7. **Green's function on the half space.** Denote

$$\mathbb{R}^N_+ = \{x = (x_1, \dots, x_N) : x_N > 0\}.$$

Reflection of $x = (x_1, \dots, x_N)$ is

$$x^* = (x_1 \dots, -x_N).$$

Let

$$\varphi^x(y) = \Phi(y - x^*) = \Phi(y - x_1, \dots, y_{N-1} - x_{N-1}, y_N + x_N),$$

where Φ is the fundamental solution. Observe that φ^x is just translation of Φ so that the singularity appears at the reflection of x. Then we have

$$\begin{cases} \Delta\varphi^x(y) = 0 & \text{if } y \in \mathbb{R}^N_+, \\ \varphi^x(y) = \Phi(y - x^*) = \Phi(y - x) & \text{if } y \in \partial\mathbb{R}^N_+, \end{cases}$$

as required in (4.13). The Green's function is therefore

$$G(x,y) = \Phi(y - x) - \varphi^x(y) = \Phi(y - x) - \Phi(y - x^*).$$

If $y \in \partial\mathbb{R}^N_+$, then $|y - x| = |y - x^*|$, $\nu = (0, \dots, 0, -1)$ and (assuming $N \geq 3$)

$$\begin{aligned} \frac{\partial G(x,y)}{\partial \nu} = DG(x,y) \cdot \nu &= -D_N G(x,y) \\ &= -D_N\Phi(y - x) + D_N\Phi(y - x^*) \\ &= c_N(2 - N)\Big(-|y - x|^{1-N}\frac{y_N - x_N}{|y - x|} + |y - x|^{1-N}\frac{y_N + x_N}{|y - x|}\Big) \\ &= \frac{1}{N\alpha(N)}\frac{2x_N}{|y - x|^N}. \end{aligned}$$

Thus the solution to

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N_+, \\ u = g & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

should be

$$u(x) = -\int_{\partial\mathbb{R}^N_+} \frac{\partial G(x,y)}{\partial \nu} g(y)\,dy = \frac{2x_N}{N\alpha(N)}\int_{\partial\mathbb{R}^N_+} \frac{g(y)}{|y - x|^N}\,dy =: \int_{\partial\mathbb{R}^N_+} K(x,y)g(y)\,dy. \tag{4.15}$$

This should also be the case for $N = 2$. Let us verify that u really is a solution.

Theorem 4.35. *If $g \in C(\partial\mathbb{R}^N_+)$ is bounded and u as in (4.15), then*

(1) *$u \in C^\infty(\mathbb{R}^N_+)$, u is bounded, $\Delta u = 0$ in $\mathbb{R}^N_+$.*
(2) *$u(x_n) \to g(x)$ whenever $x_n \to x \in \partial\mathbb{R}^N_+$.*

Proof. (1): Sketch (smoothness and details for blow through difference quotients as before):

$$\Delta u(x) = \Delta_x \int_{\partial\mathbb{R}^N_+} K(x,y)g(y)\,dy = \int_{\partial\mathbb{R}^N_+} \Delta_x K(x,y)g(y)\,dy = 0,$$

where we used that $K(x,y)$ is obviously smooth on $\partial\mathbb{R}^N_+$ if $x \in \mathbb{R}^N_+$.

(2): Let $x_0 \in \partial\mathbb{R}^N_+$, $\varepsilon > 0$, $x \in \mathbb{R}^N_+$. Then

$$
\begin{aligned}
|u(x) - g(x_0)| &= \left| \int_{\partial\mathbb{R}^N_+} K(x,y)g(y)\,dy - g(x_0) \underbrace{\int_{\partial\mathbb{R}^N_+} K(x,y)}_{=1} \right| \\
&\le \int_{\partial\mathbb{R}^N_+} K(x,y)\,|g(y) - g(x_0)|\,dy \\
&= \int_{\partial\mathbb{R}^N_+ \cap B(x_0,\delta)} K(x,y)\,|g(y) - g(x_0)|\,dy + \int_{\partial\mathbb{R}^N_+ \setminus B(x_0,\delta)} K(x,y)\,|g(y) - g(x_0)|\,dy \\
&=: I + J,
\end{aligned}
$$

where we omit the computation $\int_{\partial\mathbb{R}^N_+} K(x,y)\,dy = 1$. Then by continuity of g

$$
\begin{aligned}
I &\le \int_{\partial\mathbb{R}^N_+ \cap B(x_0,\delta)} K(x,y)\,|g(y) - g(x_0)|\,dy \\
&\le \int_{\partial\mathbb{R}^N_+ \cap B(x_0,\delta)} K(x,y)\varepsilon\,dy \le \varepsilon.
\end{aligned}
$$

Further if

$$
|x - x_0| < \delta/2, |y - x_0| \ge \delta
$$

then

$$
\begin{aligned}
|y - x_0| &\le |y - x| + |x - x_0| \\
&\le |y - x| + \frac{\delta}{2} \\
&\le |y - x| + \frac{1}{2}\,|y - x_0|,
\end{aligned}
$$

so that

$$
\frac{1}{2}\,|y - x_0| \le |y - x|.
$$

Using this

$$
\begin{aligned}
J &= \int_{\partial\mathbb{R}^N_+ \setminus B(x_0,\delta)} K(x,y)\,|g(y) - g(x_0)|\,dy \\
&\le \max_{y \in \mathbb{R}^N_+} 2\,|g(y)| \int_{\partial\mathbb{R}^N_+ \setminus B(x_0,\delta)} K(x,y)\,dy \\
&= \max_{y \in \mathbb{R}^N_+} 2\,|g(y)|\,\frac{2x_N}{N\alpha(N)} \int_{\partial\mathbb{R}^N_+ \setminus B(x_0,\delta)} |y - x|^{-N}\,dy \\
&\le c x_N \int_\delta^\infty \int_{\partial\mathbb{R}^N_+ \cap \partial B(x_0,r)} |y - x_0|^{-N}\,dS(y)\,dr \\
&= c x_N \int_\delta^\infty c r^{N-2} r^{-N}\,dr = c x_N \delta^{-1} \to 0
\end{aligned}
$$

as $x_N \to 0$. Thus we have shown that $|u(x) - u(x_0)| \le 2\varepsilon$ when $|x - x_0|$ is small enough. $\square$

4.8. **Green's function on the ball $B(0,1)$.** We know $\Phi(y-x)$ but need to solve the corrector:

$$\begin{cases} \Delta\varphi^x(y) = 0 & y \in B(0,1), \\ \varphi^x(y) = \Phi(y-x) & y \in \partial B(0,1), \end{cases}$$

to find $G(x,y) = \Phi(y-x) - \varphi^x(y)$.

We define an inversion through $\partial B(0,1)$ for $x \neq 0$

$$x^* = \frac{x}{|x|}\frac{1}{|x|} = \frac{x}{|x|^2}.$$

If $y \in \partial B(0,1)$, $x \neq 0$, then

$$\begin{aligned} |x|^2 |y-x^*|^2 &= |x|^2 (|y|^2 - 2x^* \cdot y + |x^*|^2) = |x|^2 (|y|^2 - 2\frac{x}{|x|^2} \cdot y + \frac{1}{|x|^2}) \\ &= |x|^2 - 2x \cdot y + 1 = |x-y|^2. \end{aligned} \tag{4.16}$$

Then for $y \in \partial B(0,1)$ and $x \neq 0$, $N \geq 3$,

$$\Phi(|x|(y-x^*)) = c_N |x(y-x^*)|^{2-N} = c_N |x-y|^{2-N} = \Phi(y-x)$$

and

$$\Delta_y \Phi(|x|(y-x^*)) = |x|^2 \Delta\Phi = 0$$

so that

$$\varphi^x(y) = \Phi(|x|(y-x^*)).$$

Thus

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-x^*)).$$

Also holds when $N = 2$.

Example 4.36. Consider

$$\begin{cases} \Delta u = 0 & \text{in } B(0,1), \\ u = g & \text{on } \partial B(0,1). \end{cases}$$

Then

$$u(x) = -\int_{\partial B(0,1)} g(y) \frac{\partial}{\partial \nu} G(x,y)\, dS(y),$$

with

$$\begin{aligned} D_y G(x,y) =& D_y \Phi(y-x) - D_y \Phi(|x|(y-x^*)) \\ =& c_N (2-N)(|y-x|^{1-N} \frac{y-x}{|y-x|} - ||x|(y-x^*)|^{1-N} \frac{|x|(y-x^*)|x|}{|x||y-x^*|}) \\ \overset{(4.16)}{=}& c_N (2-N)(\frac{y-x}{|y-x|^N} - \frac{|x|^2 y - x}{|y-x|^N}) \\ =& c_N (2-N) \frac{y(1-|x|^2)}{|y-x|^N}. \end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial G(x,y)}{\partial \nu} &= D_y G(y,x)\cdot \nu = D_y G(y,x)\cdot \frac{y}{|y|}\\ &= c_N(2-N)\frac{y(1-|x|^2)}{|y-x|^N}\cdot\frac{y}{|y|}\\ &= c_N(2-N)\frac{1-|x|^2}{|y-x|^N}.\end{aligned}$$

Recalling that $c_N = 1/(N(N-2)\alpha(N))$, we have arrived at the Poisson's representation formula

$$u(x) = \frac{1-|x|^2}{N\alpha(N)}\int_{\partial B(0,1)} \frac{g(y)}{|y-x|^N}\,dS(y).$$

Origin should be checked separately but we omit this.

Next we consider

$$\begin{cases}\Delta v = 0, & B(0,r),\\ v = g & \partial B(0,r).\end{cases}$$

Then $u(x) := v(xr)$ solves

$$\begin{cases}\Delta_x u(x) = r^2\Delta v(xr) = 0 & x\in B(0,1),\\ u(x) = v(xr) = g(xr) & x\in \partial B(0,1).\end{cases}$$

Thus by the previous formula

$$u(x) = \frac{1-|x|^2}{N\alpha(N)}\int_{\partial B(0,1)} \frac{g(yr)}{|y-x|^N}\,dS(y)$$

so that setting $z = xr$

$$\begin{aligned}v(z) = u(z/r) &= \frac{1-|z|^2/r^2}{N\alpha(N)}\int_{\partial B(0,1)} \frac{g(yr)}{|y-z/r|^N}\,dS(y)\\ &= \frac{1-|z|^2/r^2}{N\alpha(N)}\int_{\partial B(0,r)} \frac{g(y)}{|y-z|^N}\frac{r^N}{r^{N-1}}\,dS(y)\\ &= \frac{r^2-|z|^2/r^2}{N\alpha(N)}\int_{\partial B(0,r)} \frac{g(y)}{|y-z|^N}\,dS(y).\end{aligned}$$

4.9. **Variational method.** Consider the problem

$$\begin{cases}-\Delta u = f & \text{in } \Omega,\\ u = g & \text{on } \partial\Omega.\end{cases} \tag{4.17}$$

We show that solutions to (4.17) can be characterized as the minimizer of a certain functional. This is the so called *variational principle* sometimes also called *Dirichlet's principle*.

Definition 4.37 (Energy/variational integral)**.** The *energy functional* (or *variational integral*) corresponding to (4.17) is defined by

$$I(w) := \int_\Omega \frac{1}{2}|Dw|^2 - fw\,dx,$$

where $w \in \mathscr{A} := \left\{w\in C^2(\overline{\Omega}) : w = g \text{ on } \partial\Omega\right\}$ is the set of admissible functions.

Theorem 4.38 (Variational principle)**.** *A function $u \in C^2(\overline{\Omega})$ solves the Dirichlet problem (4.17) if and only if*

$$I(u) = \min_{w \in \mathscr{A}} I(w).$$

Proof. "$\Longrightarrow$" Observe first that for vectors $\xi, \eta \in \mathbb{R}^N$ we have

$$0 \leq |\xi - \eta|^2 = (\xi - \eta) \cdot (\xi - \eta) = |\xi|^2 - 2\xi \cdot \eta + |\eta|^2$$

so that

$$\xi \cdot \eta \leq \frac{1}{2}|\xi|^2 + \frac{1}{2}|\eta|^2. \tag{4.18}$$

Let now $w \in \mathscr{A}$. Then, integrating by parts, we obtain

$$0 = \int_\Omega (-\Delta u - f)(u - w)\,dx = \int_\Omega Du \cdot D(u-w) - f(u-w)\,dx,$$

where the boundary term vanished since $u - w = g - g = 0$ on $\partial\Omega$. Thus, using (4.18), we get

$$\begin{aligned}\int_\Omega |Du|^2 - fu\,dx &= \int_\Omega Du \cdot Dw - fw\,dx \\ &\leq \int_\Omega \frac{1}{2}|Dw|^2 + \frac{1}{2}|Du|^2 - fw\,dx,\end{aligned}$$

so that

$$I(u) = \int_\Omega \frac{1}{2}|Du|^2 - fu\,dx \leq \int_\Omega \frac{1}{2}|Dw|^2 - fw\,dx = I(w).$$

"$\Longleftarrow$" Let $\varphi \in C_0^\infty(\Omega)$. We set

$$\Psi(\varepsilon) := I(u + \varepsilon\varphi)$$

for all $\varepsilon \in \mathbb{R}$. Observe that Ψ is well defined since $u + \varepsilon\varphi \in \mathscr{A}$. Moreover, since Ψ reaches its minimum when $\varepsilon = 0$, we have $\Psi'(0) = 0$. Furthermore,

$$\begin{aligned}\Psi(\varepsilon) &= \int_\Omega \frac{1}{2}|D(u+\varepsilon\varphi)|^2 - (u+\varepsilon\varphi)f\,dx \\ &= \int_\Omega \frac{1}{2}|Du|^2 + \varepsilon Du \cdot D\varphi + \varepsilon^2|D\varphi|^2 - (u+\varepsilon\varphi)f\,dx.\end{aligned}$$

Therefore

$$\begin{aligned}0 = \Psi'(\varepsilon)\Big|_{\varepsilon=0} &= \int_\Omega Du \cdot D\varphi + 2\varepsilon|D\varphi|^2 - \varphi f\,dx\Big|_{\varepsilon=0} \\ &= \int_\Omega Du \cdot D\varphi - \varphi f\,dx \qquad \Big|\text{integrate by parts} \\ &= \int_\Omega (-\Delta u - f)\varphi\,dx\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega)$. As we have shown in the exercises, this implies $-\Delta u - f = 0$. $\square$

Remark 4.39. Existence of solutions to the Dirichlet problem could be proven by showing that the minimizer of the variational integral exists. This is done in "PDE2".

4.10. **Eigenvalue problem / Helmholz equation.** Consider $u \in C^2(\overline{\Omega})$, $\partial\Omega \in C^1$.

Definition 4.40. If the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\lambda > 0$ has a nontrivial solution (i.e. $u \not\equiv 0$), then λ is called an *eigenvalue* of Δ in Ω. The corresponding u is an *eigenfunction*.

Remark 4.41.

(1) If u is an eigenfunction, so is cu.
(2) If $\lambda \leq 0$, then there is no nontrivial solution (c.f. exercise 4, problem 6).

Definition 4.42. We define the *Rayleigh quotient*

$$Q(w) := \frac{\int_\Omega |Dw|^2 \, dx}{\int_\Omega w^2 \, dx}$$

for all admissible functions $w \in \mathscr{A}_Q := \left\{ w \in C^2(\overline{\Omega}) : w = 0 \text{ on } \partial\Omega, w \not\equiv 0 \right\}$. We also set

$$m := \inf_{w \in \mathscr{A}_Q} Q(w)$$

Remark. From Sobolev-Poincaré inequality it follows that (done in "PDE2")

$$m \geq \frac{N}{4 \operatorname{diam}(\Omega)} > 0.$$

Lemma 4.43. *If λ is an eigenvalue of Δ, then $\lambda \geq m$.*

Proof. Since

$$-\Delta u = \lambda u,$$

we have by integration by parts

$$\int_\Omega \lambda u u \, dx = \int_\Omega -\Delta u u \, dx = \int_\Omega |Du|^2 \, dx.$$

Thus

$$m \leq \frac{\int_\Omega |Du|^2 \, dx}{\int_\Omega u^2 \, dx} = \lambda. \qquad \square$$

Theorem 4.44 (Rayleigh's principle)**.** *Suppose that a minimizer of the Rayleigh quotient exists, i.e. that there is $u \in \mathscr{A}_Q$ such that*

$$Q(u) = \inf_{w \in \mathscr{A}_Q} Q(w) =: m$$

then m is the smallest eigenvalue of Δ and u is an corresponding eigenfunction.

Proof. Let $\varphi \in C_0^\infty(\Omega)$, $\varepsilon \in \mathbb{R}$ and set

$$\Psi(\varepsilon) := Q(u + \varepsilon\varphi) = \frac{\int_\Omega |D(u+\varepsilon\varphi)|^2 \, dx}{\int_\Omega (u+\varepsilon\varphi)^2 \, dx}.$$

By the assumption Ψ has a minimum at $\varepsilon = 0$ and therefore $\Psi'(0) = 0$. Thus

$$0 = \Psi'(0) = \frac{\int_\Omega 2Du \cdot D\varphi \, dx \int_\Omega u^2 \, dx - \int_\Omega |Du|^2 \, dx \int_\Omega 2u\varphi}{\left(\int_\Omega u^2 \, dx\right)^2}.$$

It follows that

$$\begin{aligned}\int_\Omega Du\cdot D\varphi\,dx &= \frac{\int_\Omega |Du|^2\,dx}{\int_\Omega u^2\,dx}\int_\Omega u\varphi\,dx\\ &= m\int_\Omega u\varphi\,dx.\end{aligned}$$

Thus by integration by parts

$$\int_\Omega -\Delta u\varphi\,dx = \int_\Omega Du\cdot D\varphi\,dx = m\int_\Omega u\varphi\,dx.$$

Since $\varphi\in C_0^\infty(\Omega)$ was arbitrary, it follows that $-\Delta u = mu$. Thus m is an eigenvalue of Δ and by Lemma 4.43 it has to be the smallest one. □

Showing that the minimizer of Rayleigh quotient exists is beyond our scope, see for example Jost: Partial differential equations. Taking the existence for granted, Theorem 4.44 then gives us the smallest eigenvalue (also called the principal eigenvalue or first eigenvalue) of Δ, which is often denoted by λ_1.

Theorem 4.45. *Let u be an eigenfunction corresponding to λ_1. Then either $u>0$ or $u<0$ in Ω.*

Proof. (Idea only; details are beyond our scope) Let u be an eigenfunction corresponding to λ_1. By continuity it suffices to show that $u(x)\neq 0$ for all x, so suppose on the contrary that $u(x_0)=0$ for some x_0. Let v be the minimizer of the Rayleigh quotient. Then by Theorem 4.44 we have

$$Q(v) = \inf_{w\in A_Q} Q(w) = \lambda_1 = Q(u), \tag{4.19}$$

where the last identity follows by integration by parts from the assumption that

$$-\Delta u = \lambda_1 u.$$

Equation (4.19) means that also u minimizes the Rayleigh quotient. The previous results hold even if the class of admissible functions A_Q is replaced by a suitable class of "weakly differentiable" functions. This way one observes that also $w := |u|$ is a minimizer of the Rayleigh quotient, since in the weak sense we have

$$Dw = \begin{cases} Du, & \text{if } u\geq 0,\\ -Du, & \text{if } u\leq 0,\end{cases}$$

so that $Q(w) = Q(|u|) = Q(u)$. By proving a Harnack's inequality for eigenfunctions, one obtains that $w\equiv 0$ in Ω. But this is against the definition of eigenfunction. □

Theorem 4.46. *The first eigenspace is one dimensional (i.e. the first eigenvalue is simple): if u and v are two eigenfunctions corresponding to λ_1, then*

$$u = kv \quad \text{in } \Omega$$

for some $k\in\mathbb{R}$.

Proof. Fix $x_0\in\Omega$ and set

$$k := u(x_0)/v(x_0).$$

Then $w := u - kv$ satisfies

$$-\Delta w = -\Delta u + k\Delta v = \lambda_1 u - k\lambda_1 v = \lambda(u-kv) = \lambda_1 w.$$

Thus, if $w \not\equiv 0$, then w is an eigenfunction corresponding to λ_1 and by Theorem 4.45 we must have $w > 0$ or $w < 0$. But neither are possible, since $w(x_0) = 0$. Thus we have $w \equiv 0$ and so $u(x) = kv(x)$ for all $x \in \Omega$. □

Theorem 4.47. *Let λ, μ be the eigenvalues corresponding to the eigenfunctions $u, v \in C^2(\overline{\Omega})$. Then either*

$$\lambda = \mu$$

or

$$\int_\Omega uv\,dx = 0.$$

That is to say, either two eigenfunctions correspond to the same eigenvalue or they are orthogonal in the Hilbert space $L^2(\Omega)$.

Proof. We have

$$\begin{aligned}
\lambda \int_\Omega uv\,dx &= -\int_\Omega \Delta uv\,dx && \Big|\text{integration by parts} \\
&= \int_\Omega Du \cdot Dv\,dx && \Big|\text{integration by parts} \\
&= -\int_\Omega u\Delta v\,dx \\
&= \mu \int_\Omega uv\,dx
\end{aligned}$$

and so $(\lambda - \mu)\int_\Omega uv\,dx = 0$. □

Remark 4.48.

(1) We state without a proof that the set of eigenvalues (spectrum) is countably infinite and discrete, i.e.

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots, \lambda_i \to \infty.$$

5. Heat equation

We consider the heat equation

$$\partial_t u = \Delta u$$

and

$$\partial_t u = \Delta u + f,$$

where $u = u(x,t)$ depends on space and time, $\partial_t u$ is time derivative and Δ is taken only respect to the space variable x:

$$\Delta u(x,t) = \sum_{i=1}^N \frac{\partial^2 u(x,t)}{\partial x_i} = \sum_{i=1}^N D_{ii}u.$$

Dirichlet problem:

$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega_T := \Omega \times (0,T), \\ u = g & \text{on } \partial_p \Omega_T, \end{cases}$$

where $\partial_p \Omega_T := \Omega \times \{0\} \cup (\partial\Omega \times [0,T])$ is called the *parabolic boundary*.

Cauchy problem:

$$\begin{cases} \partial_t u = \Delta u & \text{in } \mathbb{R}^N \times (0,T), \\ u = g & \text{on } \mathbb{R}^N. \end{cases}$$

Example 5.1. Harmonic function u, $\Delta u = 0$ is a solution $u(x,t) := u(x)$ (constant in time) $\partial_t u = 0 = \Delta u$.

Example 5.2 (Time evolution of diffusion)**.** Change is caused by the diffusion:

$$\int_U \partial_t u \, dx = \underbrace{\partial_t \int_U u \, dx}_{\text{change of amount of heat}} = \underbrace{-\int_{\partial U} F \cdot \nu \, dS}_{\text{net flux}} \overset{\text{div-thm}}{=} -\int_U \operatorname{div}(F) \, dx,$$

where ν is the exterior unit normal vector. Thus

$$\partial_t u = -\operatorname{div}(Du).$$

If again the flux density is proportional to the gradient (heat flows from hot to cold, proportional to difference)

$$F = -aDu$$

and setting for simplicity $a = 1$ we get

$$\partial_t u = -\operatorname{div}(-Du) = \Delta u.$$

If in addition, there is a heat source, change is flux plus the added heat:

$$\int_U \partial_t u \, dx = \underbrace{\partial_t \int_U u \, dx}_{\text{change of amount of heat}} = \underbrace{-\int_{\partial U} F \cdot \nu \, dS}_{\text{net flux}} + \int_U f \, dx$$

$$\overset{\text{div-thm}}{=} \int_U -\operatorname{div}(F) \, dx + \int_U f \, dx,$$

i.e.

$$\partial_t u = \Delta u + f.$$

More concretely, take 1D steel rod on $\Omega = (0,1)$ that is insulated except at the ends

$$\begin{aligned}
\partial_t u &= \partial_x \partial_x u, \\
u(x,0) &= g(x,0) \text{ initial temperature distribution} \\
u(0,t) &= g(0,t) \text{ known outside temperature at } x = 0 \\
u(1,t) &= g(1,t) \text{ known outside temperature at } x = 1.
\end{aligned}$$

Then solution $u(x,t)$ tells the temperature at x at a later time t in the rod.

5.1. **Fundamental solution.** It is known that for parabolic equations it is useful to search solutions in self-similar form: Assume

$$u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$$

and set $\lambda = t^{-1}$. Then

$$u(x,t) = t^{-\alpha} u(t^{-\beta} x, 1) =: t^{-\alpha} v(t^{-\beta} x)$$

so that we look for the solution in the form

$$u(x,t) = t^{-\alpha} v(t^{-\beta} x), \quad |x| \neq 0.$$

There are other ways to get the fundamental solution without such a guess but this is quick.

Then

$$\begin{aligned}
\partial_t u(x,t) &= -\alpha t^{-\alpha-1} v(t^{-\beta} x) - t^{-\alpha} \beta x t^{-\beta-1} \cdot Dv(t^{-\beta} x) && \Big| y = t^{-\beta} x \\
&= -\alpha t^{-\alpha-1} v(y) - t^{-\alpha-1} \beta y \cdot Dv(y)
\end{aligned}$$

and

$$\Delta u(x,t) = t^{-\alpha-2\beta}\Delta v(t^{-\beta}x) = t^{-\alpha-2\beta}\Delta v(y).$$

Plugging these into heat equation we get

$$0 = \alpha t^{-\alpha-1}v(y) + t^{-\alpha-1}\beta y\cdot Dv(y) + t^{-\alpha-2\beta}\Delta v(y).$$

We seek to simplify and select $\beta = 1/2$, thus

$$\begin{aligned} 0 &= \alpha t^{-\alpha-1}v(y) + t^{-\alpha-1}\frac{1}{2}y\cdot Dv(y) + t^{-\alpha-1}\Delta v(y) \\ \implies 0 &= \alpha v(y) + \frac{1}{2}y\cdot Dv(y) + \Delta v(y). \end{aligned}$$

To further simplify, let us look for a radial solution w such that $v(y) = w(|y|)$ (as for Laplace), so that in particular $y\cdot Dv(y) = y\cdot w'(|y|)\frac{y}{|y|}$ and recall radial Laplacian

$$0 = \alpha w(r) + \frac{1}{2}rw'(r) + w''(r) + \frac{N-1}{r}w'(r),$$

where $r = |y|$. Now, taking $\alpha = N/2$, we have

$$\begin{aligned} 0 &= \alpha w + \frac{1}{2}rw' + w'' + \frac{N-1}{r}w' \\ &= (\frac{1}{2}(r^N w) + r^{N-1}w')'r^{1-N}. \end{aligned}$$

Thus

$$\frac{1}{2}(r^N w) + r^{N-1}w' = a.$$

Assume to again simplify that $a = 0$ and thus

$$w' = -\frac{1}{2}rw$$

which has a solution

$$w(r) = ce^{-r^2/4}.$$

Recalling all the selections

$$u(x,t) = t^{-\alpha}v(t^{-\beta}x) = t^{-\frac{N}{2}}w(t^{-\frac{1}{2}}|x|) = \frac{c}{t^{N/2}}e^{\frac{-|x|^2}{4t}}.$$

Definition 5.3 (Fundamental solution to heat equation)**.** The function

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{N/2}}e^{-\frac{|x|^2}{4t}}, & (x,t)\in\mathbb{R}^N\times(0,T), \\ 0, & t\le 0 \end{cases}$$

is the fundamental solution of the heat equation.

The following lemma explains the selection of the constant.

Lemma 5.4. *For $t > 0$ we have*

$$\int_{\mathbb{R}^N}\Phi(x,t) = 1.$$

Proof. We compute

$$
\begin{aligned}
\int_{\mathbb{R}^N} \Phi(x,t) &= \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{4t}} && \Big| y = x/(4t)^{1/2}, dy = dx/(4t)^{N/2} \\
&= \frac{1}{\pi^{N/2}} \int_{\mathbb{R}^N} e^{-|y|^2}\, dy && \Big| |y|^2 = \sum_{i=1}^N y_i^2 \\
&= \frac{1}{\pi^{N/2}} \Pi_{i=1}^N \int_{-\infty}^{\infty} e^{-y_i^2}\, dy_i \\
&= 1,
\end{aligned}
$$

where we used that

$$
\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 &= \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy \\
&= \int_{-\infty}^{\infty} e^{-y^2} \int_{-\infty}^{\infty} e^{-x^2}\, dx\, dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy \\
&= \int_0^{\infty} \int_{\partial B(0,r)} e^{-r^2}\, dS\, dr \\
&= \int_0^{\infty} 2\pi r e^{-r^2}\, dr \\
&= \pi.
\end{aligned}
$$

□

5.2. **Cauchy problem.**

Theorem 5.5 (Cauchy problem for heat equation). *Let $g \in C(\mathbb{R}^N)$ be a bounded function and*

$$
u(x,t) = (\Phi * g)(x,t) = \int_{\mathbb{R}^N} \Phi(x-y,t) g(y)\, dy = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy,
$$

$t > 0$. *Then*

(1) $u \in C^2(\mathbb{R}^N \times (0,\infty))$
(2) $\partial_t u - \Delta u = 0$ *in* $\mathbb{R}^N \times (0,\infty)$
(3) *If a sequence* $(x_n,t_n) \in \mathbb{R}^N \times (0,\infty)$ *converges to* $(x_0,0)$*, then* $u(x_n,t_n) \to g(x_0)$.

Proof. (1): (Sketch) Similarly as in the proof of Theorem 4.6, we can change the order of differentiation and integration to see that

$$
\begin{aligned}
D_{ij} u = D_{ij}\Phi * g \in C(\mathbb{R}^N \times (0,\infty)), \\
\partial_t u = \partial_t \Phi * g \in C(\mathbb{R}^N \times (0,\infty)).
\end{aligned}
$$

Thus $u \in C^2(\mathbb{R}^N \times (0,\infty))$.

(2): By proof of (1) and linearity of convolution, we have

$$
\partial_t u - \Delta u = \partial_t \Phi * g - \Delta \Phi * g = (\underbrace{\partial_t \Phi - \Delta \Phi}_{=0}) * g = 0.
$$

(3): If $x_0 \in \mathbb{R}^N$, $\varepsilon > 0$, then there is $\delta > 0$ such that

$$
|g(y) - g(x_0)| < \varepsilon \quad \text{if } |y - x_0| < \delta.
$$

By Lemma 5.4 we have

$$\begin{aligned}|u(x,t)-g(x_0)| &= \left|\int_{\mathbb{R}^N}\Phi(x-y,t)(g(y)-g(x_0))\,dy\right| \\ &\le \int_{\mathbb{R}^N\cap B(x_0,\delta)}\Phi(x-y,t)\,|(g(y)-g(x_0))|\,dy \\ &\quad+\int_{\mathbb{R}^N\setminus B(x_0,\delta)}\Phi(x-y,t)\,|(g(y)-g(x_0))|\,dy \\ &=: I+J.\end{aligned}$$

Now

$$I\le\varepsilon\int_{\mathbb{R}^N\cap B(x_0,\delta)}\Phi(x-y,t)\,dy\le\varepsilon.$$

Also, like in the derivation of Green's function on plane, we observe that

$$|x-x_0|<\delta/2, |y-x_0|\ge\delta \implies |y-x|\ge\frac{1}{2}|y-x_0|.$$

Thus

$$\begin{aligned}J&\le 2\max_{\mathbb{R}^N}|g|\int_{\mathbb{R}^N\setminus B(x_0,\delta)}\Phi(x-y,t)\,dy \\ &\le\frac{c}{t^{N/2}}\int_{\mathbb{R}^N\setminus B(x_0,\delta)}e^{-\frac{|x-y|^2}{4t}}\,dy \\ &\le\frac{c}{t^{N/2}}\int_{\mathbb{R}^N\setminus B(x_0,\delta)}e^{-\frac{\left(\frac{1}{2}|y-x_0|\right)^2}{4t}}\,dy \\ &=\frac{c}{t^{N/2}}\int_{\mathbb{R}^N\setminus B(x_0,\delta)}e^{-\frac{|y-x_0|^2}{16t}}\,dy \quad \Big| z=(y-x_0)/\sqrt{t}, dz=t^{-N/2}dy \\ &=c\int_{\mathbb{R}^N\setminus B(0,\delta/\sqrt{t})}e^{-\frac{|z|^2}{16}}\,dy \\ &\to 0 \quad \text{as } t\to 0.\end{aligned}$$

Thus first choosing δ small enough and then $t>0$ small enough, we get

$$|g(y)-g(x_0)|\le I+J\le\varepsilon+J\le 2\varepsilon.$$

□

Remark 5.6.

(1) It is often denoted that

$$\begin{cases}\partial_t\Phi-\Delta\Phi=0 & \text{in } \mathbb{R}^N\times(0,\infty),\\ \Phi=\delta_0 & \text{on } \mathbb{R}^N\times\{t=0\},\end{cases}$$

where δ_0 is Dirac's delta at the origin.

(2) Observe that if $g>0$ at any point, then

$$u(x,t)=\frac{1}{(4\pi t)^{N/2}}\int_{\mathbb{R}^N}e^{-\frac{|x-y|^2}{4t}}g(y)\,dy>0$$

for any $(x,t)\in\mathbb{R}^N\times(0,\infty)$. This means that the heat equation has an infinite speed of propagation.

5.3. **Inhomogeneous Cauchy problem.** Consider

$$\begin{cases}\partial_t u - \Delta u = f & \text{in } \mathbb{R}^N \times (0,\infty),\\ u = 0 & \text{on } \mathbb{R}^N \times \{t=0\}.\end{cases}$$

We use the so called Duhamel principle and define

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^N} \Phi(x-y,t-s) f(y,s)\, dy\, ds \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\, dy\, ds. \end{aligned} \tag{5.1}$$

Theorem 5.7. *Suppose that f has compact support, f,Df, $D^2 f$, $\partial_t f$ continuous. Then for 5.1 it holds that*

(1) *u, Du, D^2u, $\partial_t u$ are continuous in $\mathbb{R}^N \times (0,\infty)$*
(2) *$\partial_t u - \Delta u = f$ in $\mathbb{R}^N \times (0,\infty)$*
(3) *If a sequence $(x_n,t_n) \in \mathbb{R}^N \times (0,\infty)$ converges to $(x_0,0)$, then $u(x_n,t_n) \to 0$.*

Proof. (1): (Sketch) We want to avoid the singularity and change variables

$$u(x,t) = \int_0^t \int_{\mathbb{R}^N} \Phi(y,s) f(x-y,t-s)\, dy\, ds$$

so that we may change the order of integration and differentiation

$$\begin{aligned} \partial_t u(x,t) &= \int_0^t \int_{\mathbb{R}^N} \Phi(y,s) \partial_t f(x-y,t-s)\, dy\, ds + \int_{\mathbb{R}^N} \Phi(y,t) f(x-y,0)\, dy, \qquad (5.2)\\ \partial_{x_i}\partial_{x_j} u(x,t) &= \int_0^t \int_{\mathbb{R}^N} \Phi(y,s) \partial_{x_i}\partial_{x_j} f(x-y,t-s)\, dy\, ds, \end{aligned}$$

and they are seen continuous using similar techniques as before. To compute the time derivative (5.2), we set $\varphi(t,s) := \int_{\mathbb{R}^N} \Phi(y,s) f(x-y,t-s)\, dy$ and observed that

$$\begin{aligned} \frac{d}{dt}\int_0^t \varphi(t,s)\, ds &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\int_0^{t+\varepsilon} \varphi(t+\varepsilon,s)\, ds - \int_0^t \varphi(t,s)\, ds \right) \\ &= \lim_{\varepsilon\to 0} \left(\int_0^{t+\varepsilon} \frac{\varphi(t+\varepsilon,s) - \varphi(t,s)}{\varepsilon}\, ds + \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \varphi(t,s)\, ds \right) \\ &= \int_0^t \frac{d}{dt}\varphi(t,s)\, ds + \varphi(t,t). \end{aligned}$$

(2): We divide the integral to the cases close and far away from the singularity:

$$\begin{aligned} \partial_t u(x,t) - \Delta u(x,t) &= \int_0^t \int_{\mathbb{R}^N} \Phi(y,s)(\partial_t - \Delta_x) f(x-y,t-s)\, dy\, ds \\ &\quad + \int_{\mathbb{R}^N} \Phi(y,t) f(x-y,0)\, dy \\ &= \int_\varepsilon^t \int_{\mathbb{R}^N} + \int_0^\varepsilon \int_{\mathbb{R}^N} + \int_{\mathbb{R}^N} \\ &= I + J + K. \end{aligned}$$

Then by Lemma 5.4

$$|J| \le \max(|\partial_t f| + |\Delta f|) \int_0^\varepsilon \int_{\mathbb{R}^N} \Phi(y,s)\, dy\, ds \le c\varepsilon.$$

Also

$$\begin{aligned}
I &= \int_\varepsilon^t \int_{\mathbb{R}^N} \Phi(y,s)(\partial_t - \Delta_x) f(x-y,t-s)\,dy\,ds \\
&= \int_\varepsilon^t \int_{\mathbb{R}^N} \Phi(y,s)(-\partial_s - \Delta_y) f(x-y,t-s)\,dy\,ds \quad \Big| \text{int. by parts, } f \text{ comp. supp.} \\
&= \int_\varepsilon^t \int_{\mathbb{R}^N} \underbrace{(\partial_s - \Delta_y)\Phi(y,s)}_{=0} f(x-y,t-s)\,dy\,ds \\
&\quad - \int_{\mathbb{R}^N} \Phi(y,t) f(x-y,0)\,dy + \int_{\mathbb{R}^N} \Phi(y,\varepsilon) f(x-y,t-\varepsilon). \\
&= -K + \int_{\mathbb{R}^N} \Phi(y,\varepsilon) f(x-y,t-\varepsilon).
\end{aligned}$$

Thus

$$I + K = \int_{\mathbb{R}^N} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\,dy$$

and

$$\begin{aligned}
\partial_t u(x,t) - \Delta u(x,t) = I + J + K &= \lim_{\varepsilon\to 0}(I + c\varepsilon + K) \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\,dy \overset{\text{c.f. Thm 5.5,ex}}{=} f(x,t).
\end{aligned}$$

(3):

$$|u(x,t)| \le \max|f| \left| \int_0^t \int_{\mathbb{R}^N} \Phi(y,s)\,dy\,ds \right| \le ct \to 0 \quad \text{as } t \to 0.$$

□

5.4. **Max principle.** Let Ω be a bounded domain, recall $\Omega_T = \Omega \times (0,T)$, $\partial_p \Omega_T = \Omega \times \{0\} \cup (\partial\Omega \times [0,T])$ and consider

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega_T, \\ u = g & \text{on } \partial_p \Omega_T. \end{cases}$$

Theorem 5.8 (Weak min/max principle, bounded set)**.** *Let Ω be a bounded set, $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$. If*

$$\partial_t u - \Delta u \ge 0 \quad \textit{(supersolution)} \tag{5.3}$$

then u attains its minimum on $\partial_p \Omega_T$, and if

$$\partial_t - \Delta u \le 0 \quad \textit{(subsolution)} \tag{5.4}$$

then u attains its maximum on $\partial_p \Omega_T$.

Proof. Assume first that u is a strict subsolution, i.e. $\partial_t u - \Delta u < 0$. Consider Ω_τ for $0 < \tau < T$. If there is $(x_0,t_0) \in \Omega_\tau \cup (\Omega \times \{t = \tau\})$ such that

$$\max_{\overline{\Omega_\tau}} u = u(x_0,t_0),$$

then

$$\partial_t u(x_0,t_0) \ge 0 \quad \text{and} \quad \Delta u(x_0,t_0) \le 0.$$

But that means

$$\partial_t u(x_0,t_0) - \Delta u(x_0,t_0) \ge 0,$$

a contradiction. Thus the maximum of u in $\overline{\Omega_\tau}$ has to be on the parabolic boundary, i.e.

$$\max_{\overline{\Omega_\tau}} u = \max_{\partial_p \Omega_\tau} u.$$

Then by continuity

$$\max_{\overline{\Omega_T}} u = \lim_{\tau \to T} \max_{\overline{\Omega_\tau}} u = \lim_{\tau \to T} \max_{\partial_p \Omega_\tau} u = \max_{\partial_p \Omega_T} u.$$

Consider then the general case $\partial_t u - \Delta u \leq 0$. Then we let

$$v_\varepsilon := u - \varepsilon t$$

so that

$$\partial_t v_\varepsilon - \Delta v_\varepsilon = \partial_t u - \Delta u - \varepsilon \leq -\varepsilon < 0,$$

i.e. v_ε is a strict subsolution. Then by first part of the proof

$$\max_{\overline{\Omega_T}} v_\varepsilon = \max_{\partial_p \Omega_T} v_\varepsilon.$$

Since $v_\varepsilon \to u$ uniformly in $\overline{\Omega_T}$, this implies that

$$\max_{\overline{\Omega_T}} u = \max_{\partial_p \Omega_T} u.$$

The proof of minimum principle for supersolutions is similar. □

Heat equation also has a mean value property when interpreted correctly.

Definition 5.9 (Heat ball)**.**

$$E(x,t,r) = \left\{ (y,s) \in \mathbb{R}^{N+1} : s < t, \Phi(x-y, t-s) > \frac{1}{r^N} \right\}.$$

Remark 5.10*.* Observe that this does not look like a ball in the usual Euclidean metric.

Theorem 5.11 (Mean value property for the heat equation)**.** *If u is a solution to the heat equation in Ω_T, then*

$$u(x,t) = \frac{1}{4r^N} \int_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} \, dy \, ds$$

for every $E(x,t,r) \Subset \Omega_T$.

We omit the proof.

This implies the strong max principle:

Theorem 5.12 (Strong max principle, bounded set)**.** *Let $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$ be a solution to the heat equation in Ω_T, and Ω bounded, connected and $(x_0, t_0) \in \Omega_T \cup (\Omega \times \{t = T\})$ such that*

$$u(x_0, t_0) = \max_{\overline{\Omega_T}} u,$$

then

$$u \equiv c \quad \textit{in} \quad \overline{\Omega_{t_0}}.$$

Remark 5.13*.* The above theorem holds also for subsolutions. Supersolutions satisfy strong minimum principle.

In a connected domain, if $u \geq 0$ is positive somewhere, then it is positive everywhere from there on: Infinite speed of propagation.

Theorem 5.14 (Uniqueness in a bounded set)**.** *Let $g \in C(\partial_p \Omega_T)$ and $f \in C(\Omega_T)$. Then the problem*

$$\begin{cases} \partial_t u = \Delta u + f & \text{in } \Omega_T, \\ u = g & \text{on } \partial_p \Omega_T \end{cases}$$

has at most one solution in $C^2(\Omega_T) \cap C(\overline{\Omega_T})$.

Proof. As before: apply maximum principle on $u - v$ and $v - u$. □

Theorem 5.15 (Max principle for the Cauchy problem)**.** *Let $u \in C^2(\mathbb{R}^N \times (0,T]) \cap C(\mathbb{R}^N \times [0,T])$ solve*

$$\begin{cases} \partial_t u = \Delta u & \text{in } \mathbb{R}^N \times (0,T), \\ u = g & \text{on } \mathbb{R}^N \times \{t = 0\}, \end{cases}$$

and satisfy the growth estimate

$$u(x,t) \le A e^{a|x|^2}, \quad (x,t) \in \mathbb{R}^N \times [0,T],$$

for some $a, A > 0$. Then

$$\sup_{\mathbb{R}^N \times [0,T]} u = \sup_{\mathbb{R}^N} g.$$

Proof. First consider the special case where

$$4aT < 1$$

so that

$$4a(T + \varepsilon) < 1 \tag{5.5}$$

for some small $\varepsilon > 0$. We fix $y \in \mathbb{R}^N$, $\mu > 0$ and define

$$v(x,t) = u(x,t) - \frac{\mu}{(T + \varepsilon - t)^{N/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}.$$

It was an excercise to show that

$$\partial_t v - \Delta v = 0 \quad \text{in } \mathbb{R}^N \times (0,T).$$

We consider $r > 0$, $\Omega := B(y,r)$ and $\Omega_T = B(y,r) \times [0,T)$. The idea is that for large enough r we have

$$v \le \sup_{\mathbb{R}^N} g \quad \text{on } \partial_p \Omega_T, \tag{5.6}$$

which lets us apply the maximum principle to get a bound for v. Clearly (5.6) holds at the bottom of the parabolic boundary, since

$$v(x,0) = u(x,0) - \frac{\mu}{(T+\varepsilon)^{N/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon)}} \le u(x,0) = g(x) \quad \text{for all } x \in \mathbb{R}^N.$$

On the other hand, if x is at the lateral part of the parabolic boundary, then $|x-y| = r$ and $0 \le t \le T$. Thus

$$\begin{aligned}
v(y,t) &= u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{N/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \\
&\le Ae^{a|x|^2} - \frac{\mu}{(T+\varepsilon-t)^{N/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}} && \Big| |x| \le |x-y| + |y| = r + |y| \\
&\le Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{N/2}} e^{\frac{r^2}{4(T+\varepsilon)}} && \Big| (5.5): \frac{1}{4(T+\varepsilon)} > a \implies \frac{1}{4(T+\varepsilon)} = a+\gamma \\
&= Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{N/2}} e^{(a+\gamma)r^2} \\
&\le \sup_{\partial_p \Omega_T} g,
\end{aligned}$$

where the last inequality follows by taking large enough r. Now we can apply the comparison principle to obtain

$$v(y,t) \le \sup_{\overline{\Omega_T}} v \le \sup_{\mathbb{R}^N} g.$$

Consequently,

$$u(y,t) = \lim_{\mu \to 0} v(y,t) \le \sup_{\mathbb{R}^N} g.$$

If the assumption $4aT \ge 1$ fails, then we repeatedly apply the previous argument in the smaller time intervals

$$[0,T'], [T',2T'], \dots,$$

where $T' = 1/(8a)$. □

Theorem 5.16 (Uniqueness to the Cauchy problem)**.** *Let $g \in C(\mathbb{R}^N)$ and $f \in C(\mathbb{R}^N \times [0,T])$. Then the problem*

$$\begin{cases} \partial_t u = \Delta u + f & \text{in } \mathbb{R}^N \times (0,T), \\ u = g & \text{on } \mathbb{R}^N \times \{t=0\}, \end{cases}$$

has at most one solution $C^2(\mathbb{R}^N \times (0,T)) \cap C(\mathbb{R}^N \times [0,T])$ satisfying the growth condition

$$|u(x,t)| \le Ae^{a|x|^2}, \quad (x,t) \in \mathbb{R}^N \times [0,T].$$

Proof. Let u and v be solutions. Then $u-v$ solves

$$\begin{cases} \partial_t(u-v) - \Delta(u-v) = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u-v = 0 & \text{on } \mathbb{R}^N \times \{t=0\}, \end{cases}$$

and

$$|u(x,t) - v(x,t)| \le 2Ae^{a|x|^2}.$$

Thus by Theorem (5.15) we have

$$u - v \le 0$$

and by a similar argument $v - u \le 0$. Thus $u = v$. □

Remark 5.17. The growth condition is essential. The problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u = 0 & \text{on } \mathbb{R}^N \times \{t=0\}, \end{cases}$$

has infinitely many solutions without the growth condition. All the solutions except $u \equiv 0$ grow fast as $|x| \to \infty$. For a counter example of Tychonov, see for example DiBenedetto: PDEs, p146.

5.5. **Energy methods and backwards in time uniqueness.** Let Ω be bounded and smooth, and consider

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega_T, \\ u = g & \text{on } \partial_p \Omega_T. \end{cases}$$

We have shown that this has only one solution, but here is another way. If u and v are solutions, $w = u - v$ solves

$$\begin{cases} \partial_t w - \Delta w = 0 & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_p \Omega_T. \end{cases}$$

Let

$$I(t) = \int_\Omega w(x,t)^2 \, dx.$$

Then

$$\begin{aligned} I'(t) = \frac{d}{dt} \int_\Omega w(x,t)^2 \, dx &= \int_\Omega \frac{d}{dt} (w(x,t)^2) \, dx \\ &= 2 \int_\Omega w(x,t) \frac{d}{dt} w(x,t) \, dx \\ &= 2 \int_\Omega w(x,t) \Delta w(x,t) \, dx && \Big| \text{int by parts} \\ &= -2 \int_\Omega |Dw(x,t)|^2 \, dx \le 0. \end{aligned}$$

Thus

$$I(t) \le I(0) = 0$$

so that $w = u - v = 0$ for all $0 \le t \le T$.

Theorem 5.18 (Backwards in time uniqueness)**.** *Let $u, v \in C^2(\overline{\Omega_T})$ solve*

$$\begin{cases} \partial_t u - \Delta u = 0 = \partial_t v - \Delta v & \textit{in } \Omega_T, \\ u = v = g & \textit{on } \partial\Omega \times [0,T] \end{cases}$$

and $u(x,T) = v(x,T)$. Then

$$u = v \quad \textit{in } \Omega_T.$$

Proof. (skip)Let $w = v - u$ and

$$I(t) = \int_\Omega w(x,t)^2 \, dx.$$

As above,

$$I'(t) = -2 \int_\Omega |Dw|^2 \, dx.$$

Then

$$
\begin{aligned}
I''(t) = -2\frac{d}{dt}\int_\Omega |Dw|^2\,dx =& -2\int_\Omega \frac{d}{dt}|Dw|^2\,dx \\
=& -4\int_\Omega Dw\cdot\frac{d}{dt}Dw\,dx \\
=& -4\int_\Omega Dw\cdot D(\frac{d}{dt}w)\,dx \\
=& 4\int_\Omega \Delta w\frac{d}{dt}w\,dx \\
=& 4\int_\Omega (\Delta w)^2\,dx.
\end{aligned}
$$

Furthermore, integrating by parts and using Hölder's inequality, we get

$$
\begin{aligned}
\int_\Omega |Dw|^2\,dx &= -\int_\Omega w\Delta w\,dx \\
&\le \int_\Omega |w|\,|\Delta w|\,dx \\
&\le (\int_\Omega |w|^2\,dx)^{1/2}(\int_\Omega |\Delta w|^2\,dx)^{1/2}.
\end{aligned}
$$

Thus

$$
(I'(t))^2 \le \left(\int_\Omega |Dw|^2\,dx\right)^2 \le 4(\int_\Omega |w|^2\,dx)(\int_\Omega |\Delta w|^2\,dx) = I(t)I''(t). \tag{5.7}
$$

If $I(t) = 0$, $0 \le t \le T$, then

$$
w = 0 \quad \text{in } \Omega_T
$$

and the claim follows. Otherwise, there exists $[t_1,t_2] \subset [0,T]$ such that

$$
\begin{cases}
I(t) > 0, & t_1 \le t < t_2, \\
I(t_2) = 0 & \text{since } w(x,T) = 0.
\end{cases}
$$

Define

$$
\Psi(t) := \log(I(t)), \quad t_1 \le t \le t_2.
$$

Then

$$
\Psi'(t) = \frac{I'(t)}{I(t)},
$$

and by (5.7)

$$
\begin{aligned}
\Psi''(t) =& \frac{I''(t)}{I(t)} - \frac{I'(t)^2}{I(t)^2} \\
\ge& \frac{I'(t)^2}{I(t)^2} - \frac{I'(t)^2}{I(t)^2} = 0.
\end{aligned}
$$

Thus Ψ is convex in (t_1,t_2). I.e.

$$
\Psi((1-\lambda)t_1 - \lambda t) \le (1-\lambda)\Psi(t_1) + \lambda\Psi(t), \quad t_1 < t \le t_2, 0 < \lambda < 1.
$$

In other notation

$$
\begin{aligned}
\log I((1-\lambda)t_1 + \lambda t) &\le (1-\lambda)\log I(t_1) + \lambda I(t) \\
&= \log I(t_1)^{1-\lambda} I(t)^\lambda.
\end{aligned}
$$

From this

$$0 \le I((1-\lambda)t_1 + \lambda t_2) \le I(t_1)^{1-\lambda} I(t_2)^{\lambda} = 0,$$

i.e.

$$I((1-\lambda)t_1 + \lambda t_2) = 0, \quad 0 < \lambda < 1,$$

so that $I(t) = 0$ for all $t_1 \le t \le t_2$, a contradiction. □

5.6. **Regularity results.**

Theorem 5.19. *Let $u \in C^2(\Omega_T)$ be a solution to the heat equation in Ω_T. Then*

$$u \in C^\infty(\Omega_T).$$

Moreover, solutions to the heat equation have derivative estimates (cf. Laplace's equation). However, solutions to the heat equation are not necessarily real analytic in t.

5.6.1. *Harnack.* We denote

$$\begin{aligned}
\tilde{Q} &= B(0,R) \times (-3R^2, 3R^2) \\
Q^+ &= B(0,R/2) \times (2R^2 - (R/2)^2, 2R^2 + (R/2)^2) \\
Q^- &= B(0,R/2) \times (-2R^2 - (R/2)^2, -2R^2 + (R/2)^2).
\end{aligned}$$

Theorem 5.20 (Harnack)**.** *Let $u \ge 0$ be a solution to the heat equation in $\tilde{Q}$. Then*

$$\sup_{Q^-} u \le c \inf_{Q^+} u,$$

where $c = c(N)$.

The proof is postponed to PDE2.

Example 5.21. "Elliptic" Harnack's inequality, i.e. where we have same cylinder on both sides does not hold in the parabolic case: the equation $\partial_t u - \partial_{xx} u = 0$ has a family of non-negative solutions in $(-R,R) \times (-R^2,R^2)$ (translated fundamental solution)

$$u(x,t) = \frac{1}{\sqrt{t+2R^2}} e^{-\frac{(x+\xi)^2}{4(t+2R^2)}},$$

where ξ is a constant. Let $x \in (-R/2, R/2)$, $x \ne 0$ and $t \in (-R^2, R^2)$. Then

$$\frac{u(0,t)}{u(x,t)} = e^{-\frac{\xi^2 - (x+\xi)^2}{4(t+2R^2)}} = e^{\frac{x^2+2x\xi}{4(t+2R^2)}} \to 0$$

as $\xi \to \infty$, for any $x < 0$. This means that the constant in an elliptic Harnack's inequality could not be independent of the solution.

6. Wave equation

We study the wave equation

$$\partial_{tt} u(x,t) = \Delta u(x,t),$$

where $u : \Omega \times (0,T) \to \mathbb{R}$, $T > 0$ and $\Omega \subset \mathbb{R}^N$ an open set.

Remark 6.1. The behavior is essentially different from the heat equation: finite speed of propagation, usually nonsmooth solutions.

Example 6.2 (Physical interpretations)**.**

$$\begin{aligned} N &= 1, \text{vibrating string,} \\ N &= 2, \text{vibrating membrane,} \\ N &= 3, \text{vibrating elastic body.} \end{aligned}$$

Let $U \subset \Omega$ be a smooth set. Then the net acceleration within U is

$$\partial_{tt} \int_U u(x,t)\,dx = \int_U \partial_{tt} u(x,t)\,dx$$

and net contact force is

$$-\int_{\partial\Omega} F \cdot \nu dS,$$

where $F = (F_1, \ldots, F_N)$ is the force caused by the oscillation. According to Newton's law "mass $\times$ acceleration = total force at the boundary" (as we assume no other forces are present, and assume mass density to be unity). Thus by divergence theorem

$$\int_U \partial_{tt} u\,dx = -\int_{\partial U} F \cdot \nu dS = -\int_U \operatorname{div} F\,dx.$$

For elastic bodies, F is a function of the displacement gradient Du, and often for small Du, the linearization $\approx -aDu$. We get

$$\partial_{tt} u = a\Delta u$$

and for simplicity we set $a = 1$ to get the wave equation.

Think about the string: it seems credible that we need

$$\begin{aligned} u(x,0) &= g(x) \text{ the initial displacement,} \\ \partial_t u(x,0) &= h(x) \text{ the initial velocity,} \end{aligned}$$

to solve the problem.

6.1. $N = 1$**, d'Alembert formula.** We study

$$\begin{cases} \partial_{tt} u - \partial_{xx} u = 0, & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x), & \text{on } \mathbb{R} \times \{t = 0\}, \\ \partial_t u(x,0) = h(x), & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

and look for an explicit solution u assuming it is smooth. Observe

$$\begin{aligned} (\partial_t + \partial_x)(\partial_t - \partial_x)u &= (\partial_t + \partial_x)\partial_t u - (\partial_t + \partial_x)\partial_x u \\ &= \partial_{tt} u + \partial_x \partial_t u - (\partial_t \partial_x u + \partial_{xx} u) \\ &= \partial_{tt} u - \partial_{xx} u. \end{aligned}$$

Denote

$$v(x,t) := (\partial_t - \partial_x)u$$

so that

$$\partial_t v + \partial_x v = 0.$$

This is a first order equation, whose solution with

$$v(x,0) = a(x)$$

is

$$v(x,t) = a(x-t).$$

Thus

$$\begin{cases} \partial_t u(x,t) - \partial_x u(x,t) = a(x-t), & \mathbb{R}\times(0,T), \\ u(x,0) = g(x), & \mathbb{R}. \end{cases}$$

This is inhomogeneous transport equation whose solution as we remember is

$$\begin{aligned} u(x,t) &= g(x+t) + \int_0^t a(-(s-t)+x-s)\,ds \\ &= g(x+t) + \int_0^t a(x+t-2s)\,ds \\ &= g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} a(y)\,dy. \end{aligned}$$

Since

$$a(x) = v(x,0) = \partial_t u(x,0) - \partial_x u(x,0) =: h(x) - g'(x),$$

we get

$$\begin{aligned} u(x,t) &= g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} h(y) \\ &= g(x+t) - \frac{1}{2}g(x+t) + \frac{1}{2}g(x-t) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy \\ &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy. \end{aligned}$$

This is d'Alembert's formula.

Remark 6.3. By d'Alembert's formula:

- If $g \in C^k$ and $h \in C^{k-1}$, then $u \in C^k$. No instant smoothening in contrast to the equation.
- The solution at (x,t) is determined by the values of g and h on $[x-t,x+t]$. Huygens principle. On the other hand every y on the initial boundary only affects on conical area: Finite speed of propagation.
- Suppose that u and v are solutions. Then $u-v$ solves the problem with zero initial values. By d'Alembert's formula this is $\equiv 0$. Uniqueness!
- Stability: let u have initial values g_1, h_1 and v have initial values g_2, h_2:

$$\begin{aligned} |u(x,t) - v(x,t)| &\le \frac{1}{2}|g_1(x+t) - g_2(x+t)| + \frac{1}{2}|g_1(x-t) - g_2(x-t)| \\ &\quad + \frac{1}{2}\int_{x-t}^{x+t} |h_1(y) - h_2(y)|\,dy \\ &\le \sup_{y\in\mathbb{R}} |g_1 - g_2| + t \sup_{y\in\mathbb{R}} |h_1 - h_2| \\ &\le \varepsilon + t\varepsilon = (1+t)\varepsilon. \end{aligned}$$

Example 6.4 (String)**.**

$$\begin{cases} \partial_{tt} u - \partial_{xx} u = 0, & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = g(x), & \text{on } \mathbb{R}\times\{t=0\}, \\ \partial_t u(x,0) = 0, & \text{on } \mathbb{R}\times\{t=0\}, \end{cases}$$

where

$$g(x) = \begin{cases} 1 - |x|^2, & -1 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Not regular, but let's apply d'Alembert's formula anyway:

$$u(x,t) = \frac{1}{2}g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy = \frac{1}{2}(g(x+t) + g(x-t)).$$

Draw pictures of $u(x,\frac{1}{2})$, $u(x,1)$ and $u(x,2)$.

6.2. **Reflection method.** Consider

$$\begin{cases} \partial_{tt}u - u_{xx} = 0, & \mathbb{R}_+ \times (0,\infty), \\ u(x,0) = g(x), & \mathbb{R}_+ \times \{t=0\}, \\ \partial_t u(x,0) = h(x) & \mathbb{R}_+ \times \{t=0\}, \\ u(0,t) = 0 & \{x=0\} \times (0,\infty). \end{cases}$$

Let us continue the functions to whole $\mathbb{R}$ by odd reflection

$$\tilde{u}(x,t) = \begin{cases} u(x,t), & x \geq 0, t \geq 0, \\ -u(-x,t), & x < 0, t \geq 0, \end{cases}$$

and define $\tilde{g}$ and $\tilde{h}$ similarly. Also assume that g,h are such that their reflections are C^2 and C^1 respectively, $g(0) = 0 = h(0)$ and $g''(0) = 0$. Then $\tilde{u}$ solves

$$\begin{cases} \partial_{tt}\tilde{u} - \tilde{u}_{xx} = 0, & \mathbb{R} \times (0,\infty), \\ \tilde{u}(x,0) = \tilde{g}(x), & \mathbb{R} \times \{t=0\}, \\ \partial_t \tilde{u}(x,0) = \tilde{h}(x), & \mathbb{R} \times \{t=0\}, \end{cases}$$

and so by d'Alembert's formula

$$\begin{aligned} \tilde{u}(x,t) &= \frac{1}{2}(\tilde{g}(x+t) + \tilde{g}(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\,dy \\ &= \begin{cases} \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy, & x \geq t \geq 0, \\ \frac{1}{2}(g(x+t) - g(t-x)) + \frac{1}{2}\int_{t-x}^{x+t} h(y)\,dy, & 0 \leq x \leq t \end{cases} \end{aligned} \tag{6.1}$$

since in the second case

$$\int_{x-t}^{x+t} \tilde{h}(y)\,dy = -\int_{x-t}^{0} h(-y)\,dy + \int_0^{x+t} h(y)\,dy = \int_{t-x}^{x+t} h(y)\,dy.$$

Example. If $h = 0$, then

$$\tilde{u}(x,t) = \begin{cases} \frac{1}{2}g(x+t) + g(x-t), & x \geq t \geq 0, \\ \frac{1}{2}(g(x+t) - g(t-x), & 0 \leq x \leq t \end{cases}.$$

Draw the pictures of $u(x,0)$, $u(x,1)$, $u(x,2)$ when

$$g(x) = \begin{cases} 1/2 + |x-1.5| & 1 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

6.3. **Spherical means.** Let $N \geq 2$ and $u \in C^2(\mathbb{R}^N \times (0,\infty))$ solve

$$\begin{cases} \partial_{tt}u - \Delta u = 0, & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = g(x), & \text{on } \mathbb{R}^N \times \{t=0\}, \\ \partial_t u(x,0) = h(x), & \text{on } \mathbb{R}^N \times \{t=0\}. \end{cases} \tag{6.2}$$

Denote

$$U(x,r,t) = \fint_{\partial B(x,r)} u(y,t)\,dS(y),$$
$$G(x,r) = \fint_{\partial B(x,r)} g(y)\,dS(y),$$
$$H(x,r) = \fint_{\partial B(x,r)} h(y)\,dS(y).$$

Fix x and regard U as a function of r,t. We will observe that then U solves the Euler-Poisson-Darboux equation (6.3) below. Observe that $\partial_{rr}U - \frac{N-1}{r}\partial_r U$ is essentially the radial Laplacian.

Lemma 6.5. *Suppose that u solves (6.2). Then it holds that $U \in C^2(\overline{\mathbb{R}_+} \times [0,\infty))$ and*

$$\begin{cases} \partial_{tt}U - \partial_{rr}U - \frac{N-1}{r}\partial_r U = 0, & \text{in } \mathbb{R}_+ \times (0,\infty), \\ U = G, \partial_t U = H, & \text{on } \mathbb{R}_+ \times \{t=0\}. \end{cases} \tag{6.3}$$

Proof. Clearly initial conditions hold:

$$U(x,r,0) = \fint_{\partial B(x,r)} u(y,0)\,dS(y) = \fint_{\partial B(x,r)} g(y)\,dS(y) = G(x,r),$$
$$\partial_t U(x,r,0) = \fint_{\partial B(x,r)} \partial_t u(y,0)\,dS(y) = \fint_{\partial B(x,r)} h(y)\,dS(y) = H(x,r).$$

When proving the mean value property for Laplacian, we obtained the formula

$$\partial_r U(x,r,t) = \frac{r}{N}\fint_{B(x,r)} \Delta u(y,t)\,dy = \frac{1}{\omega_N r^{N-1}}\int_{B(x,r)} \Delta u(y,t)\,dy,$$

so that

$$\lim_{r\to 0+} \partial_r U(x,r,t) = 0,$$

which means that $\partial_r U \in C(\overline{\mathbb{R}_+} \times [0,\infty))$. After similar computations (omitted) for $\partial_{rr}U$, we see that $U \in C^2(\overline{\mathbb{R}_+} \times [0,\infty))$.

Then

$$\begin{aligned} \partial_r(\omega_N r^{N-1} U_r) &= \partial_r\Big(\int_{B(x,r)} \Delta u(y,t)\,dy\Big) \\ &= \partial_r\Big(\int_0^r \int_{\partial B(x,\rho)} \Delta u(y,t)\,dS(y)\,d\rho\Big) \\ &= \int_{\partial B(x,r)} \Delta u(y,t)\,dS(y) \\ &= \int_{\partial B(x,r)} \partial_{tt}u(y,t)\,dS(y) \\ &= \omega_N r^{N-1}\fint_{\partial B(x,r)} \partial_{tt}u(y,t)\,dS(y) \\ &= \omega_N r^{N-1}\partial_{tt}U. \end{aligned}$$

Since

$$\partial_r(r^{N-1}\partial_r U)) = r^{N-1}\Big(\frac{N-1}{r}\partial_r U + \partial_{rr}U\Big),$$

this implies the claim. □

6.4. **Solution when $N = 3$.** Letting $r \to 0$ gives the solution to the original equation:

$$u(x,t) = \lim_{r\to 0} U(x,r,t),$$
$$h(x) = \lim_{r\to 0} H(x,r),$$
$$g(x) = \lim_{r\to 0} G(x,r).$$

Lemma 6.6. *Suppose $N = 3$ and u solves (6.2). Denote*

$$\hat{U} = rU, \quad \hat{G} = rG, \quad \hat{H} = rH.$$

Then

$$\begin{cases} \partial_{tt}\hat{U} - \partial_{rr}\hat{U} = 0 & \mathbb{R}_+ \times (0,\infty), \\ \hat{U} = \hat{G}, \partial_t \hat{U} = \hat{H}, & \mathbb{R}_+ \times \{t = 0\}, \\ \hat{U} = 0, & \{r = 0\} \times (0,\infty). \end{cases}$$

Proof. Using Lemma (6.5) and that $N = 3$, we get

$$\begin{aligned} \partial_{tt}\hat{U} = r\partial_{tt}U &= r(\partial_{rr}U + \frac{2}{r}\partial_r U) \\ &= r\partial_{rr}U + 2\partial_r U \\ &= \partial_r(U + r\partial_r \hat{U}) \\ &= \partial_r(\partial_r(\hat{U})) \\ &= \partial_{rr}\hat{U}. \end{aligned}$$

Initial conditions also hold by Lemma (6.5):

$$\begin{aligned} \hat{U}(x,r,0) &= rU(x,r,0) = rG(x,r) = \hat{G}(x,r), \\ \partial_t\hat{U}(x,r,0) &= r\partial_t U(x,r,0) = rH(x,r) = \hat{H}(x,r), \\ \hat{U}(x,0,t) &= \lim_{r\to 0} r \fint_{\partial B(x,r)} u(y,t)\,dS(y) = u(x,t) \lim_{r\to 0} r = 0. \end{aligned}$$

□

This is one dimensional wave equation so that we may use d'Alembert's formula with reflection (6.1) (a brief computation shows that $\tilde{G}, \tilde{H}$ satisfy the assumptions):

$$\hat{U}(x,r,t) = \frac{1}{2}(\hat{G}(r+t) - \hat{G}(t-r)) + \frac{1}{2}\int_{t-r}^{t+r} \hat{H}(y)\,dy, \quad 0 \le r \le t.$$

Thus

$$\begin{aligned} u(x,t) &= \lim_{r\to 0} \fint_{\partial B(x,r)} u(y,t)\,dS(y) \\ &= \lim_{r\to 0} U(x,r,t) = \lim_{r\to 0} \frac{\hat{U}(x,r,t)}{r} \\ &= \lim_{r\to 0} \left(\frac{1}{2}(\hat{G}(r+t) - \hat{G}(t-r)) + \frac{1}{2}\int_{t-r}^{t+r} \hat{H}(y)\,dy \right) \\ &= \hat{G}'(t) + \hat{H}(t) \\ &= \frac{d}{dt}(t \fint_{\partial B(x,t)} g(y)\,dS(y)) + t \fint_{\partial B(x,t)} h(y)\,dS(y) \\ &= \fint_{\partial B(x,t)} g(y)\,dS(y) + t(\frac{d}{dt} \fint_{\partial B(x,t)} g(y)\,dS(y) + \fint_{\partial B(x,t)} h(y)\,dS(y)), \end{aligned}$$

where (change variables: $y = x + tz$)

$$\begin{aligned}\frac{d}{dt}\fint_{\partial B(x,t)} g(y)\,dS(y) &= \frac{d}{dt}\fint_{\partial B(0,1)} g(x+tz)t^{N-1}t^{-(N-1)}\,dS(z)\\ &= \fint_{\partial B(0,1)} Dg(x+tz)\cdot z\,dS(z)\\ &= \fint_{\partial B(x,t)} Dg(y)\cdot\frac{y-x}{t}\,dS(y).\end{aligned}$$

Thus

$$\begin{aligned}u(x,t) &= \fint_{\partial B(x,t)} g(y)\,dS(y) + t\Big(\frac{d}{dt}\fint_{\partial B(x,t)} g(y)\,dS(y) + t\fint_{\partial B(x,t)} h(y)\,dS(y)\Big),\\ &= \fint_{\partial B(x,t)} (g(y) + Dg(y)\cdot(y-x) + th(y))\,dS(y).\end{aligned}$$

This is called the Kirchhoff formula in three dimensions.

Remark 6.7.

- Kirchoff's formula implies uniqueness and stability (c.f. remark after d'Alembert's formula)
- The value at (x,t) is determined by the values of g and h on $\partial B(x,t)$: Hyugen's principle. On the other hand, every point $y \in \mathbb{R}^3$, $t = 0$, affects the values on
$$\left\{(x,t) \in \mathbb{R}^3 \times (0,\infty) : |x-y| = t\right\}.$$
Finite speed of propagation.

6.5. **Solution when** $N = 2$**.** Assume $u \in C^2(\mathbb{R}^2 \times (0,\infty))$ and

$$\begin{cases}\partial_{tt}u - \Delta u = 0 & \text{in } \mathbb{R}^2 \times (0,\infty),\\ u(x,0) = g(x) & \text{on } \mathbb{R}^2 \times \{t=0\},\\ \partial_t u(x,0) = h(x) & \text{on } \mathbb{R}^2 \times \{t=0\}.\end{cases}$$

Define $\tilde{u} : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}$ by trivial extension:

$$\tilde{u}(x_1,x_2,x_3,t) = u(x_1,x_2,t)$$

and also define $\tilde{g}$ and $\tilde{h}$ similarly. Then

$$\begin{cases}\partial_{tt}\tilde{u} - \Delta\tilde{u} = 0 & \text{in } \mathbb{R}^3 \times (0,\infty),\\ \tilde{u}(x,0) = \tilde{g}(x) & \text{on } \mathbb{R}^3 \times \{t=0\},\\ \partial_t\tilde{u}(x,0) = \tilde{h}(x) & \text{on } \mathbb{R}^3 \times \{t=0\}.\end{cases}$$

Denote

$$x = (x_1,x_2) \in \mathbb{R}^2, \quad \tilde{x} = (x_1,x_2,0) \in \mathbb{R}^3.$$

By Kirchoff's formula

$$u(x,t) = \tilde{u}(\tilde{x},t) = \fint_{\partial B^3(\tilde{x},t)} (\tilde{g}(y) + D\tilde{g}(y)\cdot(y-\tilde{x}) + t\tilde{h}(y))\,dS(y),$$

where we integrate over the boundary of 3D ball of radius t, centered at $\tilde{x}$. Here

$$\begin{aligned}\fint_{\partial B^3(\tilde{x},t)} \tilde{g}(y)\,dS(y) &= \frac{1}{4\pi t^2}\int_{\partial B^3(\tilde{x},t)} \tilde{g}(y)\,dS(y) \quad \Big|\tilde{g}(y_1,y_2,y_3) = g(y_1,y_2)\\ &= \frac{2}{4\pi t^2}\int_{B^2(x,t)} g(y)\sqrt{1+|D\gamma(y)|^2}\,dy,\end{aligned}$$

where B^3 is a 3D ball B^2 is a 2D ball, and $\gamma : B^2(x,t) \to \mathbb{R}$,

$$\gamma(y) = \sqrt{t^2 - |y-x|^2},$$

is the parametric representation of the one half of the sphere. The factor 2 comes from the fact that sphere has two parts, upper and lower. Now

$$D\gamma(y) = -\frac{2(y-x)}{2\sqrt{t^2 - |y-x|^2}}$$

and thus

$$|D\gamma(y)| = \frac{|y-x|}{\sqrt{t^2 - |y-x|^2}}.$$

Further,

$$\sqrt{1+|D\gamma|^2} = \sqrt{1+\frac{|y-x|^2}{t^2-|y-x|^2}} = \sqrt{\frac{t^2}{t^2-|y-x|^2}} = t(t^2-|y-x|^2)^{-\frac{1}{2}}.$$

Thus

$$\fint_{\partial B^3(\tilde{x},t)} \tilde{g}\, dS = \frac{1}{2\pi t}\int_{B^2(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}\, dy = \frac{t}{2}\fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}.$$

Similarly

$$t\fint_{\partial B^3(\tilde{x},t)} \tilde{h}\, dS = \frac{t^2}{2}\fint_{B^2(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}\, dy$$

and

$$\fint_{\partial B^3(\tilde{x},t)} D\tilde{g}(y)\cdot(y-x)\, dS(y) = \frac{t}{2}\fint_{B^2(x,t)} \frac{Dg(y)\cdot(y-x)}{\sqrt{t^2-|y-x|^2}}\, dy.$$

This gives us the formula

$$u(x,t) = \frac{1}{2}\fint_{B^2(x,t)} \frac{tg(y)+t^2h(y)+tDg(y)\cdot(y-x)}{\sqrt{t^2-|y-x|^2}}\, dy$$

for the 2D problem. First solving 3D and then dropping to 2D is called the method of descent.

Remark 6.8.

- The value at (x,t) is determined by the values on $B(x,t)$ (different from 3D case). On the other hand, each point $y \in \mathbb{R}^2$, $t=0$, affects the values in the cone
$$\left\{(x,t) \in \mathbb{R}^2 \times (0,\infty) : |x-y| \le t\right\}.$$
- Also Dg present. Irregularities may focus, i.e. solution may be more irregular than the initial data.
- The above approach can be generalized to higher dimensions: solve odd N problem and then use method of descent to get to $N-1$.

6.6. **Inhomogeneous problem.** Assume $u \in C^2(\mathbb{R}^N \times (0,\infty))$ and

$$\begin{cases} \partial_{tt} u - \Delta u = f & \text{in} \mathbb{R}^N \times (0,\infty), \\ u(x,0) = 0 & \text{on } \mathbb{R}^N \times \{t=0\}, \\ \partial_t u(x,0) = 0 & \text{on } \mathbb{R}^N \times \{t=0\}. \end{cases}$$

Duhamel's principle: For $s \geq 0$, let $(x,t) \mapsto v(x,t,s)$ solve

$$\begin{cases} \partial_{tt} v(x,t,s) - \Delta v(x,t,s) = 0, & (x,t) \in \mathbb{R}^N \times (s,\infty), \\ v(x,t,s) = 0, & x \in \mathbb{R}^N \times \{t=s\}, \\ \partial_t v(x,t,s) = f(x,s) & x \in \mathbb{R}^N \times \{t=s\}, \end{cases} \tag{6.4}$$

and set

$$u(x,t) = \int_0^t v(x,t,s)\,ds.$$

Formally, this solves the inhomogeneous problem since

$$\partial_t u(x,t) = \underbrace{v(x,t,t)}_{=0} + \int_0^t \partial_t v(x,t,s)\,ds = \int_0^t \partial_t v(x,t,s)\,ds$$

and

$$\begin{aligned} \partial_{tt} u(x,t) &= \partial_t v(x,t,t) + \int_0^t \partial_{tt} v(x,t,s)\,ds \\ &= f(x,t) + \int_0^t \partial_{tt} v(x,t,s)\,ds, \end{aligned}$$

$$\Delta u(x,t) = \Delta \int_0^t v(x,t,s)\,ds = \int_0^t \Delta v(x,t,s)\,ds = \int_0^t \partial_{tt} v(x,t,s) ds$$

so that

$$\partial_{tt} u(x,t) = f(x,t) + \int_0^t \partial_{tt} v(x,t,s) = f(x,t) + \Delta u(x,t).$$

Solution to general inhomogeneous problem is then solved by $w+u$, where w is the solution to

$$\begin{cases} \partial_{tt} w - \Delta w = 0 & \text{in} \mathbb{R}^N \times (0,\infty), \\ w(x,0) = g(x) & \text{on } \mathbb{R}^N \times \{t=0\}, \\ \partial_t w(x,0) = h(x) & \text{on } \mathbb{R}^N \times \{t=0\}, \end{cases}$$

obtained by the Euler-Poisson-Darboux equation and spherical means, and u by the Duhamel's principle above.

Example 6.9. Inhomogeneous problem:

- $N = 1$: By d'Alembert's formula, equation (6.4) is solved by

$$v(x,t,s) = \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy$$

and so by Duhamel's principle

$$\begin{aligned}
u(x,t) &= \int_0^t v(x,t,s)\,ds \\
&= \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy\,ds \quad \Big| s = t-r \\
&= \frac{1}{2}\int_0^t \int_{x-r}^{x+r} f(y,t-r)\,dy\,dr \\
&= \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\,dy\,ds.
\end{aligned}$$

- $N = 3$: By Kirchoff's formula, equation (6.4) is solved by

$$v(x,t,s) = (t-s)\fint_{\partial B(x,t-s)} f(y,s)\,dS(y)$$

and so by Duhamel's principle

$$\begin{aligned}
u(x,t) &= \int_0^t v(x,t,s)\,ds \\
&= \int_0^t (t.-s)\fint_{B(x,t-s)} f(y,s)\,dS(y)\,ds \quad \Big| |\partial B(x,t-s)| = 4\pi(t-s)^2 \\
&= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)}{t-s}\,dS(y)\,ds \quad \Big| r = t-s \\
&= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\,dS(y)\,dr \\
&= \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\,dy.
\end{aligned}$$

6.7. **Energy method.**

Definition 6.10. Set

$$I(t) := \frac{1}{2}(\int_\Omega |\partial_t u(x,t)|^2 + |Du(x,t)|^2)\,dx.$$

Theorem 6.11 (Conservation of energy)**.** *Let Ω be a smooth domain and let $u \in C^2(\overline{\Omega} \times [0,T))$ solve*

$$\partial_{tt} u - \Delta u = 0$$

with $u = 0$ on $\partial\Omega \times [0,T)$. Then

$$I'(t) \equiv C.$$

Proof. The proof is a computation:

$$\begin{aligned}
I'(t) &= \frac{1}{2}\int_\Omega \partial_t(|\partial_t u|^2 + |Du|^2)\,dx. \\
&= \int_\Omega \partial_t u \partial_{tt} u + Du \cdot D\partial_t u\,dx \quad \Big| \text{int by parts} \\
&= \int_\Omega \partial_t u \partial_{tt} u - \Delta u \partial_t u\,dx \\
&= \int_\Omega \partial_t u(\partial_{tt} u - \Delta u)\,dx \\
&= 0.
\end{aligned}$$

□

Theorem 6.12 (Uniqueness by energy method)**.** *Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Then the problem*

$$\begin{cases} \partial_{tt} u - \Delta u = f & \text{in } \Omega_T, \\ u(x,0) = g(x), & \text{on } \partial_p \Omega_T, \\ \partial_t u(x,0) = h(x), & \text{on } \Omega \times \{t = 0\}, \end{cases}$$

has at most one solution $u \in C^2(\overline{\Omega} \times [0,T))$.

Proof. Let u, v be solutions and set $w = u - v$. Then w solves

$$\begin{cases} \partial_{tt} w - \Delta w = 0 & \text{in } \Omega_T, \\ w(x,0) = 0, & \text{on } \partial_p \Omega_T, \\ \partial_t w(x,0) = 0, & \text{on } \Omega \times \{t = 0\}. \end{cases}$$

Thus by conservation of energy

$$I(t) = \frac{1}{2} \int_\Omega |\partial_t w|^2 + |Dw|^2 \, dx \equiv C = I(0) = 0$$

for all $t \in [0,T]$. This means that w must be a constant in Ω_T, and since $w \equiv 0$ on $\Omega \times \{t = 0\}$, it follows that $w \equiv 0$ in Ω_T. □

Finite speed of propagation also follows from the energy method. We denote the cone

$$C = \{(x,t) : 0 \le t \le t_0, |x - x_0| \le t_0 - t\}$$

with fixed $x_0 \in \mathbb{R}^N$ and $t_0 > 0$. One can now show that external disturbances outside of C do not affect the value of u at (x_0, t_0). This follows from Kirchoff's formula, but energy method gives a more flexible proof, see Evans: PDEs, Theorem 6 on p84.

Theorem 6.13 (Finite speed of propagation)**.** *Let*

$$\partial_{tt} u - \Delta u = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty).$$

If

$$u(x,0) = 0 \quad \text{and} \quad \partial_t u(x,0) = 0$$

for all $x \in B(x_0, t_0)$, then

$$u(x,t) = 0 \quad \text{for all } (x,t) \in C.$$

7. Other ways of representing solutions

7.1. **Fourier series.** We will briefly introduce Fourier series in the space $L^2([-\pi, \pi])$. For more details on the statements below, see for example the lecture notes by Juha Kinnunen: https://math.aalto.fi/~jkkinnun/files/pde.pdf.

In this section the underlying space is $\mathbb{C}$.

Definition 7.1. The space $L^2([-\pi, \pi])$ consists of the functions $f : [-\pi, \pi] \to \mathbb{C}$ such that

$$\int_{-\pi}^{\pi} |f(t)|^2 \, dt < \infty,$$

where $|f(t)|$ denotes the modulus or length of the corresponding complex number.

Example 7.2. Let $f:[-\pi,\pi]\to\mathbb{R}$,

$$f(t)=\begin{cases}\frac{1}{\sqrt{|t|}}, & t\neq 0,\\ 0, & t=0.\end{cases}$$

Then

$$\int_{-\pi}^{\pi}|f(t)|^2\,dt=\int_{-\pi}^{\pi}1/|t|\,dt=\infty,$$

i.e. $f\notin L^2([-\pi,\pi])$.

Definition 7.3. Inner product in L^2 is defined as

$$\langle f,g\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\overline{g}(t)\,dt.$$

This induces the norm

$$\|f\|_{L^2([-\pi,\pi])}=\|f\|_{L^2}=\sqrt{\langle f,f\rangle}=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\overline{f}(t)\,dt\right)^{1/2}=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|^2\,dt\right)^{1/2},$$

where we used that for $z=(x,iy)\in\mathbb{C}$ we have $z\overline{z}=(x+iy)(x-iy)=x^2-(iy)^2=|z|^2$.

Recall the following inequalities.
Cauchy-Schwartz/Hölder:

$$|\langle f,g\rangle|\leq\|f\|_{L^2}\|g\|_{L^2}=\frac{1}{2\pi}\left(\int_{-\pi}^{\pi}|f|^2\,dt\right)^{1/2}\left(\int_{-\pi}^{\pi}|f|^2\,dt\right).$$

Triangle inequality:

$$\|f+g\|_{L^2}\leq\|f\|_{L^2}+\|g\|_{L^2}.$$

Next we denote

$$e_j:[-\pi,\pi]\to\mathbb{C},\quad e_j(t)=e^{ijt},\quad j\in\mathbb{Z}.$$

Recall *Euler's formula*

$$e_j(t)=e^{ijt}=\cos(jt)+i\sin(jt).$$

Now

$$\begin{aligned}\langle e_j,e_k\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ijt}\overline{e^{ikt}}\,dt&=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ijt}e^{-ikt}\,dt\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(j-k)t}\,dt\\&=\begin{cases}\frac{1}{2\pi}\frac{1}{i(j-k)}e^{it(j-k)}\Big|_{-\pi}^{\pi}=0 & \text{if } j\neq k,\\ 1 & \text{if } j=k.\end{cases}\end{aligned}$$

Thus $\{e_j:j\in\mathbb{Z}\}$ is an orthonormal set in $L^2([-\pi,\pi])$.

Definition 7.4. The j:th *Fourier coefficient* of $f\in L^2([-\pi,\pi])$ is

$$\hat{f}(j)=\langle f,e_j\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)e^{-ijt}\,dt,\quad j\in\mathbb{Z}.$$

The *partial sum of Fourier series* is

$$S_k(t)=S_kf(t)=\sum_{j=-k}^{k}\langle f,e_j\rangle e_j=\sum_{j=-k}^{k}\hat{f}(j)e^{ijt},\quad k=0,1,2,\ldots.$$

The *Fourier series* is the limit of the partial sum

$$\lim_{k\to\infty} S_k(t) = \lim_{k\to\infty} S_k f(t) = \lim_{k\to\infty} \sum_{j=-k}^{k} \hat{f}(j)e^{ijt} = \sum_{j=-\infty}^{j=\infty} \hat{f}(j)e^{ijt}.$$

Given $k \in \mathbb{Z}$ and coefficients $\alpha_{-k}, \dots, \alpha_k \in \mathbb{C}$, we call

$$\sum_{j=-k}^{k} \alpha_k e_j$$

a *k:th order trigonometric polynomial*. The next theorem says that the k:th partial sum of Fourier series is the best L^2-approximation of a function in the class of k:th order trigonometric polynomials.

Theorem 7.5 (Best approximation)**.** *If $f \in L^2([-\pi,\pi])$, then*

$$\|f - S_k f\|_{L^2} \le \left\| f - \sum_{j=-k}^{k} \alpha_k e_j \right\|_{L^2},$$

whenever $\alpha_k \in \mathbb{C}$.

Proof. Clearly

$$f - \sum_{j=-k}^{k} \alpha_j e_j = \left(f - \sum_{j=-k}^{k} \hat{f}(j) e_j \right) + \sum_{j=-k}^{k} (\hat{f}(j) - \alpha_j) e_j,$$

where the two functions at the right-hand side are orthogonal in $L^2([-\pi,\pi])$ since

$$\begin{aligned}
&\left\langle f - \sum_{j=-k}^{k} \hat{f}(j) e_j, \sum_{j=-k}^{k} (\hat{f}(j) - \alpha_j) e_j \right\rangle \\
&= \sum_{j=-k}^{k} (\hat{f}(j) - \alpha_j) \langle f, e_j \rangle - \sum_{j,l=-k}^{k} \hat{f}(j)(\hat{f}(l) - \alpha_l) \overbrace{\langle e_j, e_l \rangle}^{=\delta_{jl}} \\
&= \sum_{j=-k}^{k} (\langle f, e_j \rangle - \alpha_j) \langle f, e_j \rangle - \sum_{j=-k}^{k} \langle f, e_j \rangle (\langle f, e_j \rangle - \alpha_j) \\
&= 0.
\end{aligned}$$

Thus Pythagorean theorem in Hilbert spaces implies

$$\begin{aligned}
\left\| f - \sum_{j=-k}^{k} \alpha_j e_j \right\|_{L^2} &= \left\| f - \sum_{j=-k}^{k} \hat{f}(j) e_j \right\|_{L^2} + \left\| \sum_{j=-k}^{k} (\hat{f}(j) - \alpha_j) e_j \right\|_{L^2} \\
&\ge \left\| f - \sum_{j=-k}^{k} \hat{f}(j) e_j \right\|_{L^2} \\
&= \|f - S_k f\|_{L^2}.
\end{aligned}$$

□

The previous theorem implies the L^2-convergence of Fourier series.

Theorem 7.6 (L^2-convergence)**.** *If $f \in L^2([-\pi,\pi])$, then*

$$\|f - S_k f\|_{L^2} \to 0$$

as $k \to \infty$.

Proof. Trigonometric polynomials are dense in $L^2([-\pi,\pi])$ (out of scope of this course, cf. Stone-Weierstrass theorem) and so there exists a trigonometric polynomial g such that

$$\|f-g\|_{L^2} \leq \varepsilon.$$

But then by Theorem 7.5

$$\|f-S_kf\|_{L^2} \leq \|f-g\|_{L^2} \leq \varepsilon$$

provided that k is large enough. □

The previous theorem justifies writing

$$f(t) = \sum_{j=-\infty}^{\infty} \hat{f}(j)e^{ijt} \quad \text{in } L^2\text{-sense.}$$

Theorem 7.7. *Let $f \in L^2([-\pi,\pi])$ and f differentiable at t_1. Then*

$$f(t_1) = \lim_{k\to\infty} S_k(t_1)$$

in the pointwise sense.

For the proofs of the following results, see for example https://math.aalto.fi/~jkkinnun/files/pde.pdf.

Remark 7.8.

- If $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic and Lipschitz continuous, then
$$\max_{t\in[-\pi,\pi]} |S_k(t)-f(t)| \to 0, \quad \text{when } k \to \infty,$$
i.e. $S_k \to f$ uniformly.

Theorem 7.9 (Parseval)**.** *If $f \in L^2([-\pi,\pi])$, then*

$$\|f\|_2 = \sum_{j=-\infty}^{\infty} \left|\hat{f}(j)\right|^2.$$

Theorem 7.10 (Uniqueness)**.** *Let $f,g \in L^2([-\pi,\pi])$ and $\hat{f}(j) = \hat{g}(j)$, then*

$$f = g \quad \textit{in } L^2.$$

7.1.1. *Fourier series in real form (vs. complex form).* Observe

$$\begin{aligned}
S_k &= \sum_{j=-k}^{k} \hat{f}(j)e^{ijt} \\
&= \hat{f}(0) + \sum_{j=1}^{k} (\hat{f}(j)e^{ijt} + \hat{f}(-j)e^{-ijt}) \\
&= \hat{f}(0) + \sum_{j=1}^{k} (\hat{f}(j)(\cos(jt)+i\sin(jt)) + \hat{f}(-j)(\cos(jt)-i\sin(jt))) \\
&= \hat{f}(0) + \sum_{j=1}^{k} (\hat{f}(j)+\hat{f}(-j))\cos(jt) + \sum_{j=1}^{k} i(\hat{f}(j)-\hat{f}(-j))\sin(jt),
\end{aligned}$$

where

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt,$$

$$\begin{aligned}\hat{f}(j)+\hat{f}(-j) =&\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)(e^{-ijt}+e^{ijt})\,dt\\ =&\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)2\cos(jt)\,dt\\ =&\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(jt)\,dt,\end{aligned}$$

and

$$\begin{aligned}i(\hat{f}(j)-\hat{f}(-j)) &= \frac{i}{2\pi}\int_{-\pi}^{\pi} f(t)(e^{ijt}-e^{ijt})\,dt\\ &= \frac{i}{2\pi}\int_{-\pi}^{\pi} f(t)(-2i)\sin(jt)\,dt\\ &= \frac{-i^2}{\pi}\int_{-\pi}^{\pi} f(t)\sin(jt)\,dt\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(jt)\,dt.\end{aligned}$$

Thus we have obtained the following real form Fourier series

$$S_k = \sum_{j=-k}^{k} \hat{f}(j)e^{ijt} = \frac{a_0}{2} + \sum_{j=1}^{k} (a_j\cos(jt) + b_j\sin(jt)),$$

where

$$\begin{aligned}a_0 =&\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt,\\ a_j =&\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(jt)\,dt,\\ b_j =&\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(jt)\,dt.\end{aligned}$$

Remark 7.11.

- If $f:\mathbb{R}\to\mathbb{R}$, then there are only real numbers visible in the above form.
- If Fourier series is given in the real form, we can transform it back to the complex form by recalling

$$\cos(jt) = \frac{1}{2}(e^{ijt}+e^{-ijt}), \quad \sin(jt) = \frac{1}{2i}(e^{ijt}-e^{-ijt}).$$

7.1.2. *Fourier series on the general interval, odd and even functions.* If $f:[a,b]\to\mathbb{C}$, then

$$S_k(t) = \sum_{j=-k}^{k} \hat{f}(j)e^{\frac{2\pi ijt}{b-a}},$$

where

$$\hat{f}(j) = \frac{1}{b-a}\int_a^b f(t)e^{-\frac{2\pi ijt}{b-a}}\,dt, \quad j\in\mathbb{Z}.$$

This is due to the change of variables. In particular, for $f:[-L,L]\to\mathbb{C}$ in the real form

$$S_k = \frac{a_0}{2} + \sum_{j=1}^{k} (a_j\cos(\frac{\pi jt}{L}) + b_j\sin(\frac{\pi jt}{L})),$$

where

$$\begin{aligned} a_j =& \frac{1}{L}\int_{-L}^{L} f(t)\cos(\frac{\pi jt}{L})\,dt, \\ b_j =& \frac{1}{L}\int_{-L}^{L} f(t)\sin(\frac{\pi jt}{L})\,dt. \end{aligned}$$

Recall that a function $f : \mathbb{R} \to \mathbb{C}$ is odd if

$$f(-t) = -f(t)$$

and even if

$$f(-t) = f(t).$$

In the Fourier series of odd function, cos-terms vanish i.e.

$$f(t) = \sum_{j=1}^{\infty} b_j \sin(\frac{\pi jt}{L})$$

and for even function, the sin-terms vanish i.e.

$$f(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(\frac{\pi jt}{L}).$$

These are called sin- and cos-series.

Example 7.12. Consider

$$f : [-\pi,\pi] \to \mathbb{R}, \quad f(t) = \begin{cases} -1, & -\pi \le t < 0, \\ 1, & 0 \le t \le \pi. \end{cases}$$

Then the Fourier series in the real form is

$$f(t) = \frac{4}{\pi}\sum_{j=0}^{\infty} \frac{\sin((2j+1)t)}{2j+1}, -\pi < t < \pi.$$

Remark 7.13.

- Observe that Fourier series can be formed for discontinuous functions unlike Taylor's expansion.
- Fourier series have applications to PDEs as we shall see in Example 7.14 below.

7.2. **Separation of variables.**

(1) Separation of variables: In rectangular, cylindrical etc. domain separate the variables: try to find the solution in the form $v(x)w(y)$. Using this we obtain two simpler separated equations.
(2) Solve the separated equations.
(3) Solve the full problem: Look for a general solution as series. Boundary values determine the coefficients in the series.

Example 7.14. Consider

$$u : [0,a] \times [0,b] \to \mathbb{R}, \quad \Omega = (0,a) \times (0,b),$$

and

$$\begin{cases} \Delta u = \partial_{xx} u + \partial_{yy} u = 0, & \Omega \\ u(x,0) = 0, & 0 < x < a \\ u(x,b) = 0, & 0 < x < a \\ u(0,y) = 0, & 0 \leq y \leq b \\ u(a,y) = g(y), & 0 \leq y \leq b, \end{cases}$$

where $g \in C^1$.

Step 1 (separation of variables):

Set $u(x,y) = v(x)w(y)$, then

$$0 = \Delta u(x,y) = v''(x)w(y) + v(x)w''(y)$$

so that

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)}.$$

Since the left-hand side depends only x and the right-hand side only y, both sides must equal some constant $\lambda \in \mathbb{R}$. That is,

$$\begin{cases} v''(x) = \lambda v(x), & v(0) = 0 \\ -w''(y) = \lambda w(y), & w(0) = 0 = w(b). \end{cases}$$

Step 2 (Solving the separated equations): Case 1: $\lambda < 0$: $\lambda = -\mu^2$, $\mu > 0$. Now

$$w''(y) = \mu^2 w(y),$$

so that $w(y) = c_1 \sinh(\mu y) + c_2 \cosh(\mu y)$, where

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}), \quad \cosh(x) = \frac{1}{2}(e^x + e^{-x}).$$

So

$$0 = w(0) = c_2, \quad 0 = w(b) = c_1 \sinh(\mu b),$$

i.e. $c_1 = c_2 = 0$.

Case 2: $\lambda = 0$:

$$w''(y) = 0, \quad w(y) = c_1 y + c_2, \quad w(0) = 0 = w(b) \implies w = 0.$$

Case 3: $\lambda > 0$: $\lambda = \mu^2, \mu > 0$. Now

$$\begin{cases} v''(x) = \mu^2 v(x) \\ -w''(y) = \mu^2 w(y) \end{cases}$$

which gives

$$\begin{cases} v(x) = c_1 \sinh(\mu x) + c_2 \cosh(\mu x), \\ w(y) = d_1 \sin(\mu y) + d_2 \cos(\mu y). \end{cases}$$

From boundary conditions

$$v(0) = 0 \implies c_2 = 0,$$
$$w(0) = 0 \implies d_2 = 0,$$

and from $w(b) = 0$ we get that one of the two holds

$$d_1 = 0 \implies w(y) = 0 \text{ discarded}$$

or

$$\sin(\mu b) = 0 \implies \mu = \frac{j\pi}{b}, j = 1, 2, \ldots.$$

Thus

$$\begin{cases} v(x) = c_1 \sinh(\frac{j\pi x}{b}), \\ w(y) = d_1 \sin(\frac{j\pi y}{b}). \end{cases}$$

Thus

$$u_j(x,y) = v(x)w(x) = a_j \sinh(\frac{j\pi x}{b}) \sin(\frac{j\pi y}{b})$$

are nontrivial special solutions.

Step 3 (Solving the full problem): The full solution is looked for as a series

$$u(x,y) = \sum_{j=1}^{\infty} a_j \sinh(\frac{j\pi x}{b}) \sin(\frac{j\pi y}{b}). \tag{7.1}$$

The last boundary condition reads as

$$g(y) = u(a,y) = \sum_{j=1}^{\infty} a_j \sinh(\frac{j\pi a}{b}) \sin(\frac{j\pi y}{b}). \tag{7.2}$$

Extend g as an odd function to $[-b,b]$. Then its Fourier series is a sin-series

$$g = \sum_{j=1}^{\infty} b_j \sin(\frac{j\pi y}{b}), \quad \text{where}$$

$$b_j = \frac{1}{b} \int_{-b}^{b} g(y) \sin(\frac{j\pi y}{b})\, dy = \frac{2}{b} \int_0^b g(y) \sin(\frac{j\pi y}{b})\, dy. \tag{7.3}$$

Comparing the coefficients in (7.2) and (7.3), we get

$$a_j \sinh(\frac{j\pi a}{b}) = \frac{2}{b} \int_0^b g(y) \sin(\frac{j\pi y}{b})\, dy$$

$$\implies a_j = \frac{2}{\sinh(\frac{j\pi a}{b})} \int_0^b g(y) \sin(\frac{j\pi y}{b})\, dy.$$

Inserting this into (7.1) gives a representation formula for the solution. This is a formal solution at this point, as we didn't consider the convergence and regularity of the limit.

Example 7.15. Separation of variables also sometimes works for nonlinear PDEs. Consider the porous medium equation in $\mathbb{R}^N$, $N \geq 2$, $x \neq 0$,

$$\partial_t u = \Delta(u^m), \quad m > 1.$$

Try $u(x,t) = w(x)v(t)$ so that

$$v'(t)w(x) = v^m(t)\Delta w^m(x).$$

Thus

$$\frac{v'(t)}{v^m(t)} = \lambda = \frac{\Delta w^m(x)}{w(x)}.$$

A solution for $v' = \lambda v^m$ is

$$v(t) = ((1-m)\lambda t + a)^{\frac{1}{1-m}}$$

for $a \in \mathbb{R}$ that we take to be positive. Then we solve

$$\Delta w^m(x) = \lambda w(x).$$

We try $w(x) = |x|^{\alpha}$ and compute

$$\begin{aligned} Dw^m(x) &= \alpha m\,|x|^{\alpha m-2}x, \\ D^2w(x) &= \alpha m(\alpha m-2)\,|x|^{\alpha m-4}x\otimes x + \alpha m\,|x|^{\alpha m-2}I, \\ \Delta w(x) &= \alpha m(\alpha m-2+N)\,|x|^{\alpha m-2}. \end{aligned}$$

Thus

$$\begin{aligned} 0 &= \Delta w^m(x) - \lambda w(x) \\ &= \alpha m(\alpha m-2+N)\,|x|^{\alpha m-2} - \lambda\,|x|^{\alpha} && \Big|\text{choose } \alpha m-2=\alpha \\ &= \alpha m(\alpha m-2+N-\lambda)\,|x|^{\alpha} && \Big|\text{choose } \alpha m-2+N-\lambda=0 \\ &= 0, \end{aligned}$$

i.e. $\alpha = 2/(m-1)$ and $\lambda = \alpha m(\alpha m-2+N) > 0$. Thus for every $a > 0$,

$$u(x,t) = w(x)v(t) = |x|^{\alpha}\left((1-m)\lambda t + a\right)^{\frac{1}{1-m}}$$

is a solution. This represents a solution that blows up when $(1-m)\lambda t + a \to 0+$.

Example 7.16. Separation of variables also sometimes works in terms of addition instead of multiplication. Consider the Hamilton-Jacobi equation:

$$\begin{cases} \partial_t u + H(Du) = 0, & \mathbb{R}^N \times (0,\infty) \\ u(x,0) = g(x) & x \in \mathbb{R}^N, \end{cases}$$

where $H : \mathbb{R}^N \to \mathbb{R}$ is given. Try

$$u(x,t) = w(x) + v(t).$$

Then

$$0 = \partial_t u(x,t) + H(Du(x,t)) = v'(t) + H(Dw(x)).$$

Thus

$$\begin{aligned} v(t) &= -\mu t + b \\ u(x,t) &= w(x) - \mu t + b. \end{aligned}$$

In particular, if we select $w(x) = x \cdot a$ for which $H(a) = \mu$, then

$$u(x,t) = a\cdot x - H(a)t + b$$

is a solution for the initial condition $g(x) = a\cdot x + b$. Observe that in general the Hamilton-Jacobi equation is nonlinear and we cannot sum up the solutions.

8. MATLAB

8.1. **Getting started.**

- Readily installed on the university computers.
- Can be also installed on personal devices via VPN.
  - https://www.jyu.fi/digipalvelut/fi/palvelut/ohjelmistot/ohjelmistot/matlab.
- Once the command prompt is started, set "Current directory" from the pull down menu, for example "U:\Mats230.".
- If you save data on Matlab, they are now saved to this directory.
- Command `diary` will save command window input/output to a file.

8.2. **Help.**

- The following commands can be used to get help:

`doc`	open interactive manual
`lookfor`	search for keyword in all help entries
`help`	show function's help entry

- Test it yourself:

```
>> help eye
        eye(N) is the N-by-N identity matrix.
        % ... matlab displays usage of the eye
           function

>> a = eye(2)
a =
                 1      0
                 0      1
```

8.3. **Default variables.**

- Reserved variables:

`ans`	Answer of the most recent unassigned calculation
`pi`	Value of π
`i or j`	Imaginary unit
`inf`	Positive infinity
`nan`	Not-a-number

- It is possible to replace the default values of `pi`, `i` or `j` by simply setting for example `pi=2`, but this can be confusing if the standard value is also used in the same context.

8.4. **Variables.**

- = assigns to the variable on the left the value on the right as we saw above.
- Basic data type is matrix: scalar is 1×1 matrix, vectors $N \times 1$ (column) or $1 \times N$ (row).
- semicolon ; at the end of the line means that the result will not be displayed on the command prompt (comma , or nothing at the end of the line means that it is displayed).
- Pressing enter executes the command written on the line.
- **All the usual operators +,-,*,^ are matrix operations.**
- **Componentwise operations by adding a dot .*, ./, .^.**
- ' on Hermitean transpose and .' transpose (same for real matrices).
- . desimal point.
- % comment (you may write notes in the code).
- `[ ]` creating matrix, collecting block matrix, assigning multiple outputs on variables.
- `whos` Display all defined variables.
- `clear` Clear workspace variables.

```
>> a = [1 3; 2, 4]
a =
        1      3
        2      4
```

```
>> x = [5;6]
x =
     5       6

>> x2 = a*x
x2 =
    23      34

>> x2*x

% Matlab displays error because * was used  with
   incorrect dimensions for matrix multiplication.

ans =
    28      40

>> pi*a
ans =
        3.1416     9.4248
        6.2832    12.5664
```

- Plenty of ready functions: For example `sin`, `cos`, `asin`, `sqrt`, `exp`, `log`, `abs`, `mod`.
- See also: `help elfun`.

```
>> y = sin(x); %Compute sin from elements of x.
>> y'
ans =
   -0.9589    -0.2794
>> x2 .* x
ans =
        115
        204
>> x2' * x
ans =
   319
>> a*[1 + 3i; 2.5]
ans =
        8.5000 + 3.0000i
        12.0000 + 6.0000i
```

8.5. Indexing.

- Matrix indexing: you can pick elements using brackets. **Indexing begins with 1 in Matlab, not with zero.**
- : alone creates a vector with consecutive integers or any specified interval. In indexing means all the entries in a row or column.

```
>> a = [1 4 7; 2 5 9; 3 6 9]
a =
```

```
     1     4     7
     2     5     9
     3     6     9

>> a(1, 2) %pick element at row 1, column 2.
ans =
     4

>> a(1, :) %pick the first row
ans =
     1     4     7

>> a(:, 2) %pick the second column
ans =
     4      5      6

>> a(:, 2:end) %pick columns starting from the
   second column until the end
ans =
     4     7
     5     9
     6     9

>> b = 1:3 %make a column vector with consecutive
   integers 1, 2, 3.
b =
     1     2     3

>> a(b, [1,3]) %take columns 1 and 3  with rows in
   b
ans =
     1     7
     2     9
     3     9

>> b = [1, 3];
>> a(b, [1,3])
ans =
        1       7
        3       9

>> a(1:2,1:2) = eye(2) %Set a sub-matrix of a equal
    to the 2x2 identity matrix.
a =
        1     0     7
        0     1     9
        3     6     9
```

- Summary:

`a=[1,2,3]`	Set the variable a to a column vector $(1,2,3)$.
`[1,2,3;4,5,6]`	Matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.
`[X;Y]`	Block matrix $\begin{bmatrix} X & Y \end{bmatrix}$.
`2:5`	A vector $(2,3,4,5)$.
`1:3:10`	A vector $(1,4,7,10)$.
`a(2)`	2nd element of a vector.
`a(1,2)`	$(1,2)$:th element of a matrix.
`a(1,2)`	First row of a matrix.
`a(:,3)=b`	Set the third column of a matrix to the value `b`.
`a(3:2;end,:)`	Matrix formed by every second rows from the third to the last row of `a`

8.6. Elementary matrix operations.

- Some common operations:

`linspace`	Linearly spaced vector.
`eye`	Identity matrix.
`diag`	Diagonal matrix or a diagonal of a matrix.
`rand`	Random matrix with elements uniformly distributed in $(0,1)$.
`zeroes`	Matrix of zeroes.
`ones`	Matrix of ones.
`length`	Length of a matrix.
`size`	Size of a matrix.
`repmat`	Replicate matrix.
`find`	Find nonzero elements.

- See also `help elmat`.
-

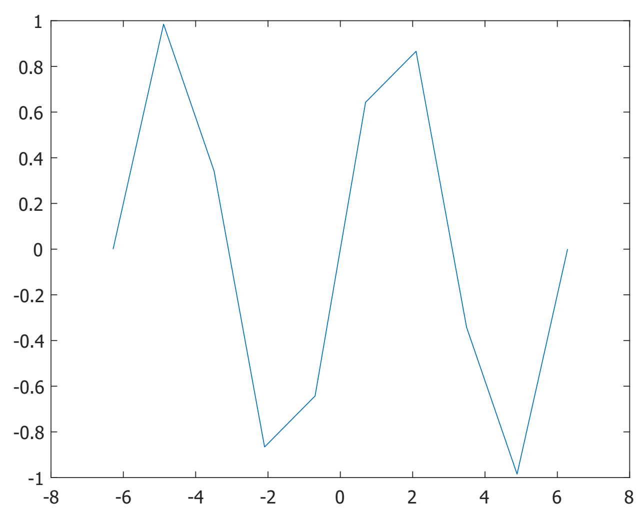
```
>> a =
    0.8147    0.9134    0.2785
    0.9058    0.6324    0.5469
    0.1270    0.0975    0.9575

>> b = ones(3,1)
b =
        1
        1
        1

>> c = zeros(1, 3)
c =
        0      0      0

>> d = [a b;c 42]
d =
        0.8147    0.9134    0.2785    1.0000
        0.9058    0.6324    0.5469    1.0000
        0.1270    0.0975    0.9575    1.0000
```

```
        0          0          0          42.0000

>> diag(d)
ans =
        0.8147
        0.6324
        0.9575
        42.0000

>> e = diag([pi exp(1)])
e =
    3.1416       0
        0     2.7183
```

8.7. **Graphics.**

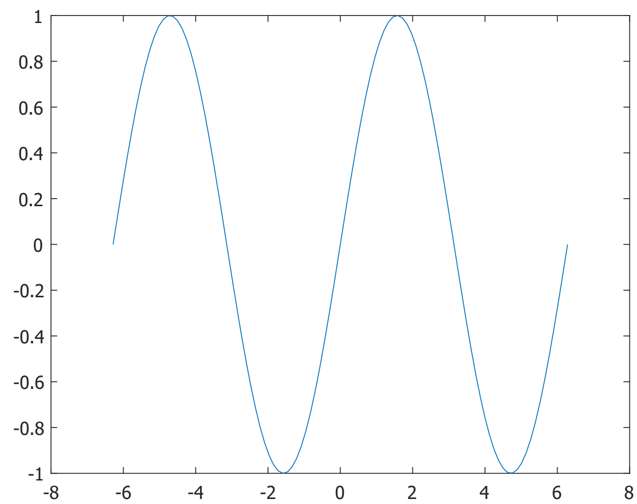
- Matlab can be used for visualization.

```
>> x = linspace(-2*pi, 2*pi, 10);
>> y = sin(x);
>> plot (x,y)
```



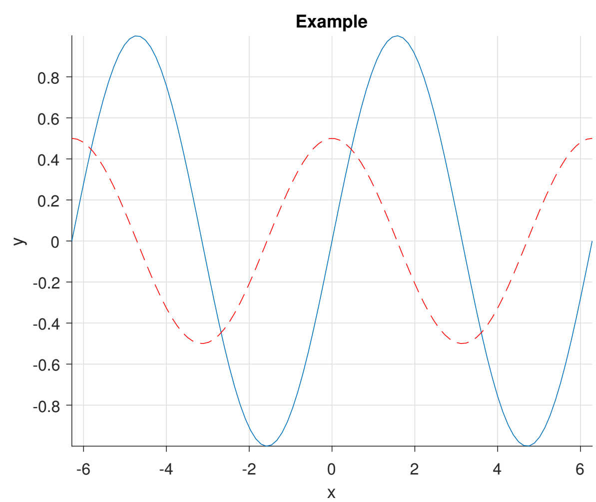
- Let's use finer spacing for x variable:

```
>> x = linspace(-2*pi, 2*pi, 100);
>> y = sin(x);
>> plot (x,y)
```



- Lets add another curve and customize settings:

```
>> hold on %holds the current plot so that
   subsequent graphing commands add to the existing
   graph.
>> plot(x, 0.5*cos(x), 'r--')
>> axis tight, grid on, box off
>> xlabel 'x'; ylabel 'y'; title 'Example'
>> set(gca, 'tickdir', 'out');
>> print -depsc example.svg
```



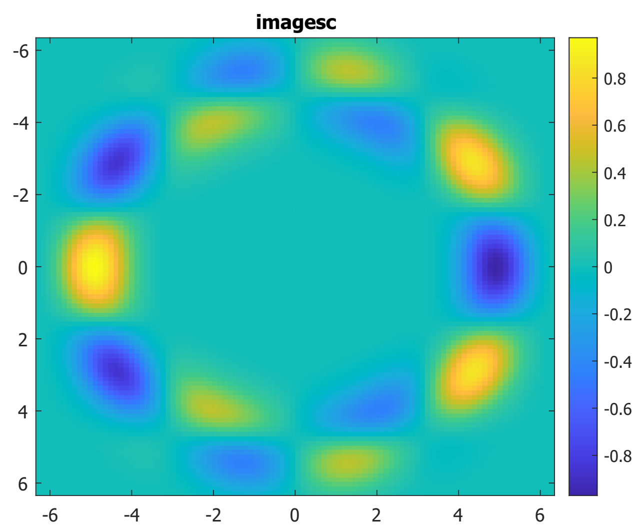
- For visualization of 2D functions we use `meshgrid` command.

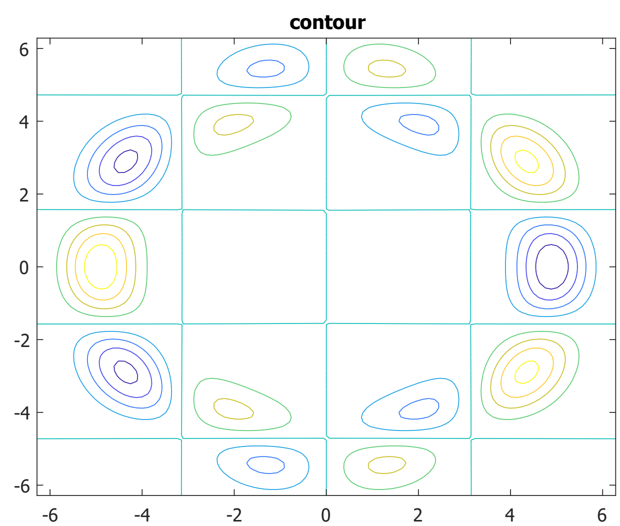
```
>> x = [0 1 2 3];
>> y = [8 9 10];
>> [X, Y] = meshgrid(x, y)
X =
         0         1         2         3
         0         1         2         3
         0         1         2         3
Y =
```

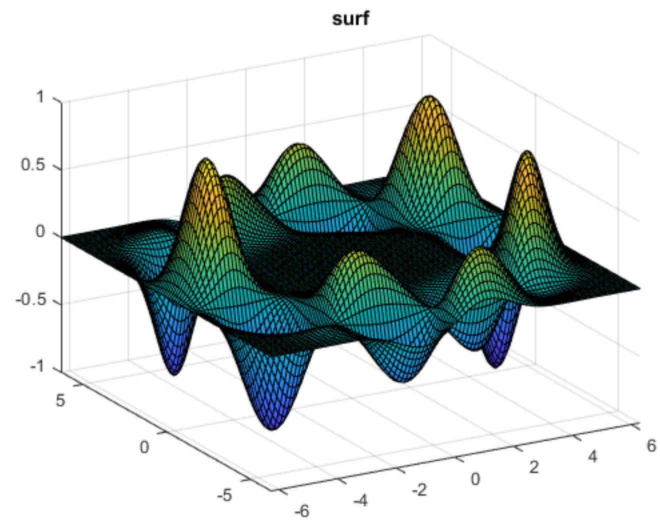
```
        8         8         8         8
        9         9         9         9
        10        10        10        10
```

- Using this:

```
>> x = linspace(-2*pi, 2*pi, 100); y=x;
>> [X, Y] = meshgrid(x, y)
>> f = sin(X) .* cos(Y) .* exp(-(sqrt(X .^2 + Y
   .^2) - 5) .^2);
>> imagesc(x, y, f); colorbar; title 'imagesc'
>> contour(x, y, f); title 'contour'
>> surf(x, y, f); title 'surf'
```







8.8. **M-files and own functions.**

- You can create "M-files" for example `mfile.m` that includes a list of commands.
- Another type of "M-file" is a function which means that you can call it with parameters and it returns values.
- You can execute a file by calling its name without .m, so in this case `mfile`.
- Function is saved in a file with the **same name**. For example, if our function is called `test`, we create `test.m`.
- To create a function, type `edit test` on the command prompt. Then type for example the following function:

```
%test.m
function d = test(x, y) %function name has to
   correspond to the filename
d = x + y;
```

Now you can call the function from the command prompt:

```
>> test(1, 2)
ans =
         3

>> test([1, 1], [1, 2])
ans =
         2      3
```

- It is also possible to define a function directly using the `@` symbol.

```
>> g = @(x,y) x + y;
>> g(1,2)

ans =
     3
```

8.9. Loops and logical expressions.

- Loops and logical expressions can be used in Matlab:

`if`	a section code is executed if a condition is true. A condition is true if it equals 1.
`for`	a section of code is executed while an index goes through a set of values.
`while`	a section of code is executed while a condition is true.

- In logical conditions you may use for example <, <=, ==, &, |, see `help relop`.
-

```
N = 6;
a = eye(N);
for k=1:N %execute code for k=1, 2, ..., N.
    j = 1;
    while j<k %execute code while j<k
            if mod(k, j) == 0
                    a(k, j) = 1;
            end %closes if
            j = j+1;
        end %closes while
end %closes for
```

- Logical operations also work for matrices and can be used for indexing.

```
>> a= 1:5, b = rand(1,5)*5
a =
     1     2     3     4     5
b =
    4.0736    4.5290    0.6349    4.5669    3.1618

>> a < b
ans =
        1    1    0    1    0

>> find(a < b)
ans =
        1     2     4

>> a(a < b) = 0

a =
     0     0     3     0     5
```

8.10. Programming strategy.

- For loops are slow in Matlab. If possible, replace them with built-in matrix operations.
- Not like this:

```
n = 1100;
x = linspace(0, 2*pi, n);
y = linspace(0, 3*pi, n);
```

```
for i=1:n %nested for loops create mesh, slow.
        for j=1:n
                X(i, j) = x(i);
                Y(i, j) = y(j);
                Z(i, j) = sin(x(i)) * cos(y(j));
        end
end
mesh(X, Y, Z)
```

- But like this:

```
n = 1100;
x = linspace(0, 2*pi, n);
y = linspace(0, 3*pi, n);

[X, Y] = meshgrid(x, y) %create mesh by calling the
    meshgrid function, fast.
Z = sin(X) .* cos(Y).
mesh(X, Y, Z)
```

8.11. Saving and loading data.

- Variables can be saved by the command `save` and loaded by `load`.

```
>> a = 1:5;
>> save test
>> clear
>> a
% Matlab shows error : unrecognized function or
   variable 'a'.

>> load test
>> a
a =
        1     2     3     4     5
```

- You can also load data produced by other programs. For example in Excel you can save data in CSV-form. Read data in Matlab using `csvread` or `dlmread`.

9. Numerics

We will briefly describe a method that can be used to discretize and obtain approximate solutions to the Poisson's equation. Convergence proofs for numerical approximations (i.e. proofs that the numerical method actually provides something close to a true solution) require regularity estimates etc. and are beyond our score.

9.1. Laplace equation.

9.1.1. *1D case.* Consider the equation

$$\begin{cases} \Delta u = u'' = f, & \text{in } (0,1) \\ u(0) = a, u(1) = b. \end{cases} \tag{9.1}$$

Let $u : \mathbb{R} \to \mathbb{R}$ be a smooth function. By Taylor's theorem

$$\begin{aligned} u(x+h) =& u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \frac{1}{6}u'''(x)h^3 + O(h^4), \\ u(x-h) =& u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 - \frac{1}{6}u'''(x)h^3 + O(h^4). \end{aligned}$$

Subtracting we get

$$u'(x) = \frac{1}{2h}(u(x+h) - u(x-h)) + O(h^2)$$

and summing up

$$u''(x) = \frac{1}{h^2}(u(x-h) - 2u(x) + u(x+h)) + O(h^2).$$

Next we divide the interval $[0,1]$ into

$$x_j = jh, \quad h = \frac{1}{m+1}, \quad j = 0, 1 \dots, m+1,$$

and denote by u_j the approximation for $u(x_j)$ that we are searching for. Thus, dropping the error terms

$$\Delta u(x) = u''(x) \approx \frac{1}{h^2}(u(x+h) - 2u(x) + u(x-h)). \tag{9.2}$$

and so 9.1 can be discretized as

$$\frac{1}{h^2} \begin{pmatrix} -h^2 & 0 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & -h^2 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_m \\ u_{m+1} \end{pmatrix} = \begin{pmatrix} a \\ f_1 \\ \vdots \\ f_m \\ b \end{pmatrix},$$

where we have encoded the boundary values on the first and last row and $f_j = f_j(x_j)$. In the case of zero boundary values, we have $0 = a = b = u_1 = u_2$ and so the above system reduces to

$$\Delta_h \overline{u} = \overline{f},$$

where $\Delta_h \in \mathbb{R}^{M \times M}$, $\overline{u}, \overline{f} \in \mathbb{R}^m$ are as follows

$$\Delta_h = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}, \overline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \quad \text{and} \quad \overline{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}. \tag{9.3}$$

We want to solve $\overline{u}$ and know $\overline{f}$ so

$$\overline{u} = \Delta_h^{-1} \overline{f},$$

where $\Delta^{-1}_{h,D-D}$ is the inverse of the matrix Δ_h. If we have nonzero Dirichlet condition $u(0) = a$, we get from (9.2)

$$u''(h) = \frac{1}{h^2}(a - 2u(h) + u(2h)) = \frac{1}{h^2}(-2u(h) + u(2h)) + \frac{a}{h^2}$$

and moving a/h^2 into $\overline{f}$,we can still write the equation in the form $\Delta_h u = (f_1 - a/h^2, f_2, \ldots)^T$.

9.1.2. *2 dimensional case.* Consider the equation

$$\begin{cases} \Delta u(x) = f(x) & \text{in } \Omega = (0,1) \times (0,1). \\ u(x) = g & \text{on } \partial\Omega. \end{cases} \tag{9.4}$$

Set

$$m_1, m_2 \in \mathbb{N}, \quad h_1 = \frac{1}{m_1+1}, \quad h_2 = \frac{1}{m_2+1}.$$

Now it is convenient to write the approximate values $u(ih_1, jh_2) \approx u_{i,j}$ as

$$U := U_{h_1,h_2} = \begin{pmatrix} u_{1,1} & \ldots & u_{1,m_2} \\ \vdots & \ddots & \vdots \\ u_{m_1,1} & \ldots & u_{m_1,m_2} \end{pmatrix}, \quad i = 1, \ldots, m_1, \quad j = 1, \ldots, m_2$$

Approximating like in the 1D-case, we have

$$\begin{aligned} \partial_{x_1x_1} u(ih_1, jh_2) &\approx \frac{1}{h_1^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}), \\ \partial_{x_2x_2} u(ih_1, jh_2) &\approx \frac{1}{h_2^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}). \end{aligned} \tag{9.5}$$

Assuming that u has zero boundary values (i.e. $u_{0,j} = u_{j,0} = 0$ for all i, j), the above can be written in matrix form

$$\begin{aligned} \partial_{x_1x_1} U &\approx \Delta_{h_1} U, \\ \partial_{x_2x_2} U &\approx U \Delta^T_{h_2}, \end{aligned}$$

where Δ_{h_1} is the matrix defined at (9.3). Thus we get a discretization for the 2-dimensional Laplacian

$$\Delta u \approx \Delta_{h_1,h_2} U := \Delta_{h_1} U + U \Delta^T_{h_2}. \tag{9.6}$$

We want to solve for U and therefore wish to write $\Delta_{h_1,} U + U \Delta^T_{h_2,}$ in the form matrix * vector since this is then easy to solve by inverting the matrix. To this end, let us denote the vector

$$T(U) = (u_{1,1}, \ldots, u_{m_1,1}, u_{1,2}, \ldots, u_{m_1,m_2})^T \in \mathbb{R}^{m_1 m_2} \tag{9.7}$$

i.e. write the matrix U as a vector by putting the columns at the top of each other (In Matlab simply $U(:)$).

Lemma 9.1. *Let $A \in \mathbb{R}^{m_1 \times m_2}$, $U \in \mathbb{R}^{m_1 \times m_2}$, $B \in \mathbb{R}^{m_2 \times m_2}$. Then*

$$T(AUB^T) = (B \otimes A) T(U)$$

where

$$B \otimes A = \begin{pmatrix} b_{11}A & b_{12}A & \ldots & b_{1m_1}A \\ b_{21}A & b_{22}A & \ldots & b_{2m_2}A \\ \vdots & \vdots & & \vdots \\ b_{m_21}A & b_{m_22}A & \ldots & b_{m_2m_2}A \end{pmatrix}.$$

Proof. Write down carefully both sides. □

Using the lemma, we have by (9.6)

$$
\begin{aligned}
T(\Delta_{h_1,h_2}U) &= T(\Delta_{h_1}U + U\Delta_{h_2}^T) \\
&= T(\Delta_{h_1}UI_{m_2}^T + I_{m_1}U\Delta_{h_2}^T) && \Big|\text{lemma} \\
&= I_{m_2}^T \otimes \Delta_{h_1}T(U) + \Delta_{h_2}^T I_{m_1}T(U) \\
&= (I_{m_2} \otimes \Delta_{h_1} + \Delta_{h_2} \otimes I_{m_1})T(U),
\end{aligned}
$$

where $I_{m_1} \in \mathbb{R}^{m_1 \times m_1}$, $I_{m_2} \in \mathbb{R}^{m_2 \times m_2}$ are identity matrices. In the case $h = h_1 = h_2$, we have $m_1 = m_2$ and obtain from above

$$
\begin{aligned}
T(\Delta_{h_1,h_2}U) &= (I_m \otimes \Delta_h + \Delta_h \otimes I_m)T(U) \\
&= \left(\begin{pmatrix} \Delta_h & 0 & \dots & 0 \\ 0 & \Delta_h & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Delta_h \end{pmatrix} + \begin{pmatrix} (\Delta_h)_{11}I & (\Delta_h)_{12}I & \dots & (\Delta_h)_{1m}I \\ (\Delta_h)_{21}I & (\Delta_h)_{22}I & \dots & I \\ \vdots & \vdots & \ddots & \vdots \\ (\Delta_h)_{m1}I & (\Delta_h)_{m1}I & \dots & (\Delta_h)_{mm}I \end{pmatrix} \right) T(U) \quad \Big|\text{def. of } \Delta_h \\
&= \left(\begin{pmatrix} \Delta_h & 0 & \dots & 0 \\ 0 & \Delta_h & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Delta_h \end{pmatrix} + \frac{1}{h^2} \begin{pmatrix} -2I & I & 0 & 0 & \dots & 0 \\ I & -2I & I & 0 & \dots & 0 \\ 0 & I & -2I & I & \cdots & 0 \\ 0 & 0 & I & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & -2I & I \\ 0 & 0 & 0 & 0 & I & -2I \end{pmatrix} \right) T(U) \\
&= \frac{1}{h^2} \begin{pmatrix} X & I & 0 & 0 & \dots & 0 \\ I & X & I & 0 & \dots & 0 \\ 0 & I & X & I & \cdots & 0 \\ 0 & 0 & I & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & X & I \\ 0 & 0 & 0 & 0 & I & X \end{pmatrix} T(U) =: AT(U),
\end{aligned}
\tag{9.8}
$$

where

$$
X = h^2\Delta_h - I = \begin{pmatrix} -4 & 1 & 0 & 0 & \dots & 0 \\ 1 & -4 & 1 & 0 & \dots & 0 \\ 0 & 1 & -4 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & -4 & 1 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{pmatrix}.
$$

On the otherhand, since u solves 9.8, we should have

$$
T(\Delta_{h_1,h_2}U) \approx T(F),
$$

where

$$
F = \begin{pmatrix} f_{1,1} & \dots & f_{1,m_2} \\ \vdots & \ddots & \vdots \\ f_{m_1,1} & \dots & f_{m_1,m_2} \end{pmatrix}, \quad f_{i,j} = f(ih_1, jh_2).
$$

Thus $T(U)$ can be solved from 9.8 by inverting the matrix A, i.e.

$$
T(U) = A^{-1}T(\Delta_{h_1,h_2}U) = A^{-1}T(F).
$$

If the boundary data is not zero, it has to be taken into account in the approximation 9.5. We have

$$\begin{aligned}
\partial_{x_1x_1}u(h_1,jh_2) &\approx \frac{1}{h_1^2}(u_{i+1,j}-2u_{i,j}+u_{i-1,j}) \\
&= \frac{1}{h_1^2}(u_{i+1,j}-2u_{i,j}+g(0,jh_2), \\
\partial_{x_1x_1}u(m_1h_1,jh_2) &\approx \frac{1}{h_1^2}(g(1,jh_2)-2u_{ij}+u_{i-1,j}), \\
\partial_{x_2x_2}u(ih_1,h_2) &\approx \frac{1}{h_2^2}(u_{i,j+1}-2u_{i,j}+u_{i,j-1}) \\
&= \frac{1}{h_2^2}(u_{i,j+1}-2u_{i,j}+g(ih_1,0)), \\
\partial_{x_2x_2}u(ih_1,m_2h_2) &\approx \frac{1}{h_2^2}(g(ih_1,1)-2u_{ij}+u_{i,j-1}).
\end{aligned}$$

Moving g to the left-hand side and writing in matrix form, we obtain this time

$$\begin{aligned}
-G+\partial_{x_1x_1}U &\approx \Delta_{h_1}U, \\
-G+\partial_{x_2x_2}U &\approx \Delta_{h_2}^T U,
\end{aligned}$$

where

$$G=\begin{pmatrix} g_{1,1} & \cdots & g_{1,m_2} \\ \vdots & \ddots & \vdots \\ g_{m_1,1} & \cdots & g_{m_1,m_2}\end{pmatrix}, \quad g_{i,j}=\begin{cases} h_1^{-2}g(0,jh_2), & i=1, \\ h_1^{-2}g(1,jh_2), & i=m_1, \\ h_2^{-2}g(ih_1,0), & j=1, \\ h_2^{-2}g(ih_1,1), & j=m_2, \\ 0, & \text{otherwise.}\end{cases}$$

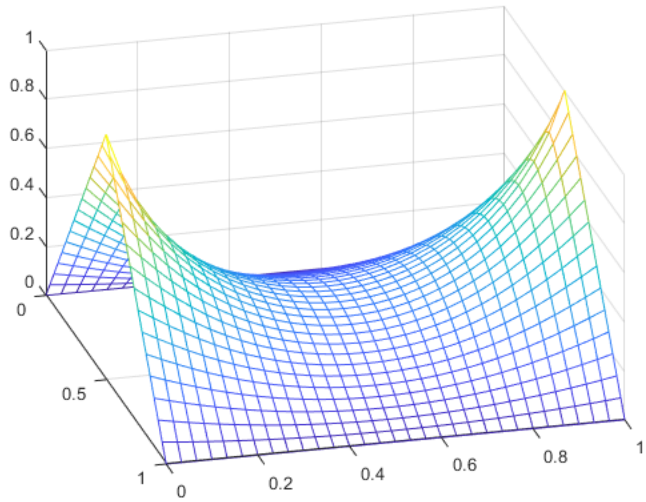
Thus we can do the same computations as in the case of zero boundary values, and obtain that

$$T(U)=A^{-1}T(\Delta_{h_1,h_2}U)=A^{-1}(T(F-G))=:A^{-1}T(b). \tag{9.9}$$

Example 9.2. Let us solve the equation

$$\begin{cases} \Delta u = f(x,y) & \text{in } (0,1)\times(0,1), \\ u=g(x,y) & \text{on } \partial(0,1)\times(0,1), \end{cases}$$

where $g(x,y)=1-|1-2x|$ and $f(x,y)=0$.




```
%solver.m

%the problem data
g = @(x, y) (1 - abs(1 - 2*x));
f = @(x, y) 0;

m1=30;
m2=30;
h1 = 1/(m1+1);
h2 = 1/(m2+1);

%We begin by creating the matrices D_h_1 and D_h_2
   defined in (9.3).
%The diag(v, k) command creates a matrix whose k:th
   diagonal is the vector v. (the main diagonal
   corresponds to k = 0).
D1 = (-diag(2 * ones(m1, 1)) + diag(ones(m1-1, 1), 1) ...
          + diag(ones(m1-1, 1), -1)) / h1^2;

D2 = (-diag(2 * ones(m2, 1)) + diag(ones(m2-1, 1), 1) ...
          + diag(ones(m2-1, 1), -1)) / h2^2;

%We create the matrix A as in computation (9.7). The
   Kronecker tensor product, denoted by \otimes in the
   lecture note, is achieved by kron function. Sparse
   format saves memory for matrices with many zeros.
A = sparse(kron(eye(m2), D1) + kron(D2, eye(m1)));

%Next we create the matrix b in (9.9).
b = zeros(m1, m2);
for i=1:m1
for j=1:m2
b(i, j) = f(i*h1, j*h2) ...
```

```
          - (g(0, j*h2) * (i == 1) ...
             + g(1, j*h2) * (i == m1))/ h1^2 ...
          - (g(i*h1, 0) * (j == 1) ...
             + g(i*h1, 1) * (j == m2))/ h2^2;
end
end

%Reshape the matrix to vector format as defined in (9.7)
   .
Tb = reshape(b, m1*m2, 1);

%Solve TU from the equation (9.9). One could also write
   TU = inv(A)*Tb, but the following implementation is
   recommended for solving equations.
TU=A\Tb;

%Reshape the solved function into matrix format.
U = reshape(TU, m1, m2);

%To plot the function, we must still include the
   boundary values.
u = zeros(m1+2, m2+2);
u(2:m1+1, 2:m2+1) = U; %interior
u(:, 1)           = g(0:h1:1, 0); %u(i, 1) = g(i, 0)
u(:, m2+2)        = g(0:h1:1, 1);
u(1, :)           = g(0, 0:h2:1);
u(m1+2, :)        = g(1, 0:h2:1);

mesh(linspace(0,1,m1+2), linspace(0,1,m2+2), u');
```