Metriset avaruudet, Exercise set 2
14.9. 2022

1. Let d_{SNCF} be the metric on $\mathbb{R}^2$ from the first problem of the first exercise set. Describe the balls $B(x,1)$ with respect to this metric when $x \neq 0$.

2. Show that each closed ball in a metric space is a closed set and that each set that only contains a single point (a one-point set) is also closed.

3. Show that each finite intersection of open sets is open.

4.

a) Show that E is open if and only if $E = \text{int}E$. Here $\text{int}E$ is the collection of all x for which there exists $r > 0$ so that $B_d(x,r) \subset E$, the set of interior points of E.

b) Show that E is closed if and only if $E = \overline{E}$. Here $\overline{E}$ is by definition the intersection of all closed sets that contain E.

5.

a) Show that

$$\overline{E} = E \cup \partial E = \text{int}E \cup \partial E.$$

Here ∂E is the collection of those points in X that are neither interior points of E nor interior points of $X \setminus E$.

b) Show that $\partial(X \setminus E) = \partial E$.

6. Show that E is closed if and only if $\partial E \subset E$.

7. The discrete metric gives us examples of (X,d) for which

$$\overline{B(x,r)} \neq \overline{B}(x,r)$$

for some x and $r > 0$. Give a similar example where $X = A \subset \mathbb{R}^2$ and d is the metric induced by the Euclidean metric of $\mathbb{R}^2$.

8. Let (X,d) be a metric space and $0 < \alpha < 1$. Set

$$d'(x,y) := d(x,y)^\alpha$$

when $x, y \in X$. Show that also d' is a metric.

Hint: show first that $(a+b)^\alpha \leq a^\alpha + b^\alpha$ when $a, b \geq 0$ e.g. by

1) by first verifying the inequality when $a + b = 1$

2) and then the general case via the special case.