Metriset avaruudet, Exercise set 3
21.9. 2022

1. Is the identity mapping id, $id(x) = x$, continuous as a map $(\mathbb{R}^2, d_E) \to (\mathbb{R}^2, d_{SNCF})$ when d_E is the usual Euclidean metric on $\mathbb{R}^2$ and d_{SNCF} as in the first problem of the previous exercise set, so continuous with respect to the given metrics? What about the inverse map?

2. Let $(V, ||.||)$ be a normed space and d the metric induced by the norm. Show that $n : (V, d) \to (\mathbb{R}^1, d_E)$, $n(v) := ||v||$, is continuous when d_E is the Euclidean metric on $\mathbb{R}$.

3. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ be a mapping. Show that f is continuous if and only if $f^{-1}(F)$ is closed for each closed set $F \subset Y$.

4. Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. Suppose that $f : X \to Y$ and $g : Y \to Z$ continuous with respect to the (respective) given metrics. Show that also $g \circ f : X \to Z$ is continuous with respect to the relevant metrics.

5. Let (X, d) be a metric space and fix $x_0 \in X$. Let $a \in X$. Define $\phi_a : X \to \mathbb{R}$ by setting

$$\phi_a(x) := d(x, x_0) - d(x, a).$$

Show that ϕ_a is continuous with respect to the metric d on X and the Euclidean metric d_E on $\mathbb{R}$. Further show that ϕ_a is a bounded map i.e. that $\phi_a(X)$ is a bounded subset of $\mathbb{R}$.

6. Let (X, d) be a metric space and fix $x_0 \in X$. Set $K(a) = \phi_a$, where ϕ_a is the map defined in the preceding problem. Then $K : X \to C^0_{raj}(X, \mathbb{R})$ by the conclusions of the preceding problem. Show that K is an isometric embedding with respect to the metrics d ja d_∞.

7. Let $f : X \to Y$ be a homeomorphism with respect to the metric d_X on X and d_Y on Y. Let $E \subset X$. Show that $f(\partial E) = \partial(f(E))$.

8. Let (X, d) be a metric space. Show that there is metric d' for which (X, d') is a bounded metric space for which the identity mapping $id : (X, d) \to (X, d')$ is a homeomorphism.