Metriset avaruudet, Excercise set 4
28.9. 2022

1. Let $(X, d_X),\ (Y, d_Y)$ be metric spaces and $C \subset X,\ D \subset Y$ closed sets. Show that $C \times D$ is closed in the product space $X \times Y$ with respect to the natural induced metrics, see VII.5.

2. Let $(X, d_X),\ (Y, d_Y)$ be metric spaces $E \subset X,\ F \subset Y$. Determine $\partial(E \times F)$ with respect to the natural metrics on $X \times Y$.

3. Show that each converging sequence is a Cauchy sequence.

4. Show that each Cauchy sequence is bounded.

5. Give an example of a sequence $(x_k)_k$ of points on the real line so that the sequence is not a Cauchy sequence but, nevertheless,

$$\lim_{k \to \infty} |x_k - x_{k+1}| = 0.$$

6. Show that each uniformly continuous map maps Cauchy sequences to Cauchy sequences.

7. Let δ be the discrete metric on $X = \mathbb{R}^2$. Which sequences are Cauchy sequences? Explain.

8. Consider the norms

$$||f||_1 := \int_0^1 |f(t)|dt$$

and

$$||f||_\infty := \max_{t \in [0,1]} |f(t)|$$

on $X = C^0([0, 1])$. Show that these two norms are not equivalent. How is this possible?