Metriset avaruudet, Exercise set 5
5.10. 2022

1. Let (X, d_X), (Y, d_Y) be metric spaces. Show that the product space $X \times Y$ is complete with respect to the usual induced metric (d_1, d_2 or d_∞) if and only if both X and Y are complete.

2. Show that the metric space $(\mathbb{R}^n, d_E)$ is complete when $n \geq 2$ using the fact that $(\mathbb{R}, |.|)$ is complete.

3. Give an example of homeomorphic metric spaces (X, d_X) ja (Y, d_Y) so that the former one is complete but the latter one is not complete.

4. Let (X, d) be a complete metric space and let

$$X \supset E_1 \supset E_2 \supset \ldots$$

be nested non-empty closed sets. Assume that

$$\mathrm{diam}(E_j) \to 0$$

when $j \to \infty$. Show that

$$\cap_{j=1}^{\infty} E_j = \{x_0\}$$

for some $x_0 \in X$.

5. Let (X, d_X) be a non-empty metric space $E \subset X$. Show that E is dense if and only if $E \cap U \neq \emptyset$ for each non-empty open set U.

6. Consider $\mathbb{R}$ equipped with the discrete metric. Is $\mathbb{Q} \subset \mathbb{R}$ dense? Why?

7. Let $E \subset X$ be dense with respect to d_X and let $f : (X, d_X) \to (Y, d_Y)$ continuous and onto. Show that $f(E)$ is dense in Y. Do we need the assumption that f is onto? What about continuity?

8. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with respect to the usual Euclidean metrics. Show that the graph

$$G(f) = \{(x, f(x)) : \; x \in \mathbb{R}\}$$

of f is a closed set in $\mathbb{R}^2$ with respect to d_E. Hint: consider the mapping $F(x, y) := y - f(x)$.