**Metriset avaruudet, Excercise set 7**
**19.10. 2022**

**1.** Let $(X, d)$ be a metric space, $E \subset X$ be connected, and let $x \in \partial E$. Show that $E \cup \{x\}$ is connected.

**2.** Explain the necessary modifications to your proof for Problem 1 to prove: if $E$ is connected and $E \subset A \subset \overline{E}$, then also $A$ is connected.

**3.** Show that a metric space $(X, d)$ is connected if and only if $\emptyset$ and $X$ are the only subsets of $X$ that are both open and closed.

**4.** Show that $\mathbb{R}^n \setminus \{0\}$ equipped with the Euclidean metric is pathwise connected when $n \geq 2$.

**5.** Let $U \subset \mathbb{R}^n$ be non-empty and open with respect to the Euclidean metric. Fix $x_0 \in U$ and let $V$ the collection of those points of $U$ that can be joined to $x_0$ with a path in $U$, and $W$ the collection of those points of $U$ that cannot be joined to $x_0$ with a path in $U$. Show that both $V$ and $W$ are open.

**6.** Show that every connected open set in the Euclidean space $\mathbb{R}^n$ is path-connected.

**7.** We say that a path $\gamma : [a, b] \to \mathbb{R}^n$ is a broken line if there exist $k \geq 2$ and $a = a_0 < a_1 < ... < a_k = b$ so that the restriction of $\gamma$ to each $[a_j, a_{j+1}]$ is of the form $\gamma(t) = x_j + \frac{t-a_j}{a_{j+1}-a_j}(x_{j+1} - x_j))$. Use the previous problem to show that all pairs of points in a given open, non-empty connected set of $\mathbb{R}^n$ (with respect to the Euclidean metric) can be joined by a broken line.

**8.** Explain why the infimum of the lengths of the broken lines joining given points $x, y$ gives a metric in a non-empty, connected, open subset of the Euclidean space $\mathbb{R}^n$. Notice that the length of a broken line can be computed since it is piecewise $C^1$.