Metriset avaruudet, Exercise set 1
7.9. 2022

1. Show that d_{SNCF} is a metric on $\mathbb{R}^2$. Recall that $d_{SNCF}(x, y) = ||x - y||$ if both x, y belong to the same line through the origin (i.e. are linearly dependent) and $d_{SNCF}(x, y) = ||x|| + ||y||$ otherwise. Here $||.||$ is the usual Euclidean norm.

2. Show that d defined as follows is a metric on $\mathbb{R}^2$: let $d(x, y) = 0$ when $x = y$ and $d(x, y) = ||x|| + ||y||$ otherwise. Can you give a geometric interpretation for this metric?

3. Show that

$$||x||_1 := |x_1| + |x_2|$$

defines a norm on $\mathbb{R}^2$. Here $x = (x_1, x_2)$.

4. Show that

$$||x||_\infty = \max\{|x_1|, |x_2|\}$$

defines a norm on $\mathbb{R}^2$.

5. Let d_1 be the metric given by the norm $||.||_1$ from problem number 3 and similarly d_∞ for the norm d_∞ from problem number 4. Draw the sets

$$A = \{x \in \mathbb{R}^2 : d_1(x, (1, 1)) < 1\}$$

and

$$C = \{x \in \mathbb{R}^2 : d_\infty(x, (1, 1)) \leq 1\}.$$

6. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by the formula

$$f(x_1, x_2, x_3) = (x_1 + 1, 2x_2, 3x_3)$$

and let d_E be the Euclidean metric on $\mathbb{R}^3$. Set

$$d(x, y) := d_E(f(x), f(y))$$

when $x, y \in \mathbb{R}^3$. Show that d is a metric.

7. Let the couples (X, d_X) and (Y, d_Y) be metric spaces: X is a set and d_X is a metric on X, and d_Y on Y, respectively. Set

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

when $(x_1, y) \in X \times Y$ ja $(x_2, y_2) \in X \times Y$. Show that d is a metric on $X \times Y$. How is this related to problem number 3?

8. Let (X, d) be a metric space. Set $d'(x, y) = \min\{d(x, y), 1\}$ when $x, y \in X$. Show that also d' is a metric on X.